Philosophy 231—Introduction to Deductive Logic

Informal Proof of the irrationality of $\sqrt{2}$.

- Suppose $\sqrt{2}$ is rational, i.e., $\sqrt{2} = \frac{m}{n}$, for some integers m and n such that $\frac{m}{n}$ is a non-reducible fraction.

  - First suppose that m is odd.

    $\sqrt{2}n = m$ (multipying both sides by n).

    $2n^2 = m^2$ (squaring both sides).

    So m^2 is even (i.e., equal to some integer multiple of 2).

    So m is even, since (odd $\times$ odd) = odd.

    $\perp$ (Contradiction.)

  - Now suppose that m is even, i.e., $m = 2k$ for some integer k.

    $\sqrt{2}n = m$ (multipying by n).

    $2n^2 = m^2 = (2k)^2 = 4k^2$ (squaring both sides).

    $n^2 = 2k^2$ (divide both sides by 2).

    So n^2 is even (i.e., equal to some integer multiple of 2).

    So n is even (since (odd $\times$ odd) = odd).

    So $\frac{m}{n}$ is reducible (since numerator and denominator are both even).

    $\perp$ (Contradiction.)

$\Rightarrow$ So $\sqrt{2}$ is not rational.