Cornell University
Department of Economics

Econ 620 - Spring 2003
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Solution to the Midterm

Exercise 1:

We are told that $y'y = 760; 1'y = 30; x'y = 1300; 1'x = 0; 1'1 = 31$ and $x'x = 2480$.
Therefore, $n = 1'1 = 31$ and $\bar{x} = \frac{1}{n}1'x = 0$. This was worth 5 points each.

Let $A = I_n - 1(1'1)^{-1}1'$ (this matrix transforms to deviations from the mean). We can also calculate $y'Ay = y'y - n\bar{y}^2 = 760 - 31(\frac{30}{31})^2 = 730.97$ as well as $x'Ax = x'x - n\bar{x}^2 = 2480$ and also $y'Ax = y'x - n\bar{x}\bar{y} = 1300$.

In a regression with only one right hand side variable apart from the intercept (the simple regression case), we know that the R^2 is the square of the correlation between x and y (see lecture 3). Therefore, $R^2 = \frac{1300^2}{730.97 \cdot 2480} = 0.932$. (this part, 5 points)

The estimator for the slope parameter is $\hat{\beta} = \frac{\sum_i (y_i - \bar{y})(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2} = \frac{y'Ax}{x'Ax} = \frac{1300}{2480} = \frac{130}{248} = 0.524$. (this part, 5 points)

The t-statistic is $T = \frac{\hat{\beta} - \beta}{se(\hat{\beta})}$ and under $H_0; T = \frac{\hat{\beta}}{se(\hat{\beta})}$. So we need to find the standard error of the slope estimator. We can calculate it as $se(\hat{\beta}) = \sqrt{\frac{s^2}{\sum_i (x_i - \bar{x})^2}}$ where $s^2 = \frac{e'e}{29}$. In order to find e'e, note that $e'e = TSS - ESS = TSS - R^2\ TSS = (1 - R^2)TSS = (1 - R^2)y'Ay = 49.52$ (this part, 5 points). Hence, $s^2 = 1.707$ (this part, 1 point), $se(\hat{\beta}) = 0.026$ (this part, 2 points) and $T = \frac{\hat{\beta}}{se(\hat{\beta})} = 19.98$ (this part, 2 points).

When we regress x on y, for the reason mentioned above, the R^2 is the same (5 points) and $\hat{\beta}^* = \frac{\sum_i (y_i - \bar{y})(x_i - \bar{x})}{\sum_i (y_i - \bar{y})^2} = \frac{y'Ax}{y'Ay} = \frac{1300}{730.97} = 1.778$ (5 points). Indeed, from problem 5 in homework 2 you can check that $\hat{\beta}\hat{\beta}^* = R^2$.

Again, the t-statistic is $T = \frac{\hat{\beta}^* - \beta^*}{se(\hat{\beta}^*)}$ and under $H_0; T = \frac{\hat{\beta}^*}{se(\hat{\beta}^*)}; e^{*\prime}e^* = (1 - R^2)TSS^* = 0.932\ 2480 = 168$ (5 points), so $s^2 = \frac{168}{29} = 5.79$ (1 point), $se(\hat{\beta}^*) = \sqrt{\frac{5.79}{730.97}} = 0.089$ (2 points) and $T = \frac{\hat{\beta}^*}{se(\hat{\beta}^*)} = 19.98$ (why is it the same number?)

In both cases, we would reject the null hypothesis at a significance level of 1 %.

Exercise 2:

The dimensions are: V is $n \times n$, X is $n \times k$, so VX is $n \times k$ and R is $k \times k$, since $R = (X'V^{-1}X)^{-1}X'X$. In the case where $k = 1$; X is an eigenvector of V. In general, if the columns of X are each linear combinations of the same k eigenvectors of V, then $\hat{\beta}_{OLS} = \hat{\beta}_{GLS}$. This is hard to check and would usually be a bad assumption. Note also that there is no hope to estimate V since there are $\frac{n(n+1)}{2}$ parameters to estimate and we only have n observations.

In the SURE system, the dimensions are : V is $mn \times mn$, X is $mn \times mk$ and R is $mk \times mk$. Using the notation as in lecture 12, $V = \Sigma \otimes I_n$ and $X = I_m \otimes X_1$. Note that Σ is $m \times m$ and X_1 is $n \times k$. By property of Kronecker products, $V^{-1} = \Sigma^{-1} \otimes I_n$. Therefore,

$$\begin{aligned}
R &= (X'V^{-1}X)^{-1}X'X \\
&= [(I_m \otimes X_1')(\Sigma^{-1} \otimes I_n)(I_m \otimes X_1)]^{-1}(I_m \otimes X_1')(I_m \otimes X_1) \\
&= [(\Sigma^{-1} \otimes X_1'X_1)]^{-1}(I_m \otimes X_1'X_1) \\
&= [\Sigma \otimes (X_1'X_1)^{-1}](I_m \otimes X_1'X_1) \\
&= \Sigma \otimes I_k
\end{aligned}$$

The condition is easily verified: $VX = (\Sigma \otimes I_n)(I_m \otimes X_1) = \Sigma \otimes X_1$ for the LHS and for the RHS, $XR = (I_m \otimes X_1)(\Sigma \otimes I_k) = \Sigma \otimes X_1$.

Grading policy: the first part of the question was worth 10 points and checking for the SURE system was worth 15 points.

Exercise 3:

To find c, we need to use the fact that the density has to integrate to one and be nonnegative. Therefore, since the density of each random variable is $f(x_i) = ce^{-|x_i|/\gamma}$, $i = 1, 2, ..., N$, each with support the real line, we have that

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x_i)dx_i &= \int_{-\infty}^{+\infty} ce^{-|x_i|/\gamma}dx_i \\
&= \int_{-\infty}^{0} ce^{x_i/\gamma}dx_i + \int_{0}^{+\infty} ce^{-x_i/\gamma}dx_i \\
&= c\left[\int_{-\infty}^{0} e^{x_i/\gamma}dx_i + \int_{0}^{+\infty} e^{-x_i/\gamma}dx_i\right] \\
&= c\left[\gamma e^{x_i/\gamma}\Big|_{-\infty}^{0} - \gamma e^{-x_i/\gamma}\Big|_{0}^{\infty}\right] \\
&= c[\gamma + \gamma] \\
&= 2\gamma c
\end{aligned}$$

Hence, $c = \frac{1}{2\gamma}$.

Since the density of each random variable is $f(x_i) = \frac{1}{2\gamma}e^{-|x_i|/\gamma}$, $i = 1, 2, ..., N$, each with support the real line, it follows that the joint density of the N independent random variables is the product of them. Therefore, the likelihood function is

$$L(\gamma) = \left(\frac{1}{2\gamma}e^{-|x_1|/\gamma}\right)...\left(\frac{1}{2\gamma}e^{-|x_n|/\gamma}\right) = \left(\frac{1}{2\gamma}\right)^N e^{-\frac{1}{\gamma}\sum_{i=1}^{N}|x_i|}$$

and the log likelihood,

$$\ell(\gamma) = -N\{\ln 2 + \ln \gamma\} - \frac{1}{\gamma}\sum_{i=1}^{N}|x_i|$$

Therefore, the score equation is $\ell'(\theta) = -\frac{N}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{N} |x_i| = 0$ and the MLE is

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{N} |x_i|}{N}$$

By noting that $\ell''(\theta) = \frac{N}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{N} |x_i|$; and using (from the score equation) that $\frac{1}{\theta^2}\sum_{i=1}^{N} |x_i| = \frac{N}{\theta}$, it follows that the information matrix is $\mathcal{I}(\theta) = E(-\ell''(\theta)) = \frac{N}{\theta^2}$ and therefore, $\mathcal{I}^{-1}(\theta) = \frac{\theta^2}{N}$.

By property of MLE estimators, the asymptotic distribution of the MLE estimator of θ is normal with mean θ and asymptotic variance equal to $\frac{\theta^2}{N}$; i.e.: $\hat{\theta}_{MLE} \overset{a}{\sim} N(\theta, \frac{\theta^2}{N})$

Grading policy: the correct c is worth 5 points, the MLE estimator 10 points, and for the correct asymptotic distribution 10 points.

Statistics:

Lower Quartile: 52
Median: 64
Upper Quartile: 90