Cornell University
Department of Economics

Econon 620 - Spring 2002
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Solution to the Midterm

Exercise 1:
(this part, 12 points) We have the model

$$y_i = \alpha + \varepsilon_i$$

with $\mathrm{E}(\varepsilon_i) = 0$ and $\mathrm{Var}(\varepsilon_i) = \sigma^2 x_i$ and the $\varepsilon_i's$ are uncorrelated. To find the BLUE for α, we need to transform the model (use GLS), since the errors are heteroskedastic (OLS will not give us a BLUE estimator for α because the errors do not have covariance matrix $\sigma^2 I$). Dividing by $\sqrt{x_i}$(this can be done because x_i are strictly positive), the transformed model is

$$\frac{y_i}{\sqrt{x_i}} = \alpha \frac{1}{\sqrt{x_i}} + \varepsilon_i \frac{1}{\sqrt{x_i}}$$

and note that the errors in this tranformed model have expectation zero, are uncorrelated and have variance σ^2. Applying OLS to the transformed model, we have that the sum of squares residuals to be minimized with respect to a is:

$$S(a) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (\frac{y_i}{\sqrt{x_i}} - a\frac{1}{\sqrt{x_i}})^2 = \sum_{i=1}^{n} \frac{1}{x_i}(y_i - a)^2$$

The first order condition to this minimization problem is,

$$(-2)\sum_{i=1}^{n} \frac{1}{x_i}(y_i - \widehat{\alpha}) = 0$$

Second order conditions for this minimization problem are satisfied since $S''(a) = S''(\widehat{\alpha}) = 2 > 0$.
Hence,

$$\widehat{\alpha} = \frac{\sum_{i=1}^{n} \frac{y_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

(this part, 8 points)
Substituing y_i for $\alpha + \varepsilon_i$, we can write

$$\widehat{\alpha} = \frac{\sum_{i=1}^{n} \frac{\alpha+\varepsilon_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}} = \alpha + \frac{\sum_{i=1}^{n} \frac{\varepsilon_i}{x_i}}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

Therefore, $\mathrm{E}(\widehat{\alpha}) = \alpha$ and

$$
\begin{aligned}
Var(\widehat{\alpha}) &= E(\widehat{\alpha}-\alpha)^2 \\
&= E\left[\frac{\sum_{i=1}^{n}\frac{\varepsilon_i}{x_i}}{\sum_{i=1}^{n}\frac{1}{x_i}}\right]^2 \\
&= \left[\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i}}\right]^2 E(\sum_{i=1}^{n}\frac{\varepsilon_i}{x_i})^2 \\
&= \left[\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i}}\right]^2 \sum_{i=1}^{n}\frac{E(\varepsilon_i^2)}{x_i^2} \\
&= \left[\frac{1}{\sum_{i=1}^{n}\frac{1}{x_i}}\right]^2 \sum_{i=1}^{n}\frac{\sigma^2 x_i}{x_i^2} \\
&= \frac{\sigma^2}{\sum_{i=1}^{n}\frac{1}{x_i}}
\end{aligned}
$$

Grading policy: for the correct estimator 4 out of 12 points, and for the correct variance 4 out of 8 points were assigned. If you did not get the correct formula, points were assigned to the procedure.

Exercise 2:

Since the density of each random variable is f(t_i) $= \gamma e^{-\gamma t_i}$, $\quad i = 1, 2, ..., N$, each with support the nonnegative reals, it follows that the joint density of the N independent random variables is the product of them. Therefore, the likelihood function is

$$L(\gamma) = (\gamma e^{-\gamma t_1})...(\gamma e^{-\gamma t_N}) = \gamma^N e^{-\gamma \sum_{i=1}^{N} t_i}$$

and the log likelihood,

$$\ell(\gamma) = N \log_e \gamma - \gamma \sum_{i=1}^{N} t_i$$

Therefore, the score equation is $\ell'(\gamma) = \frac{N}{\gamma} - \sum_{i=1}^{N} t_i = 0$ and the MLE is

$$\widehat{\gamma}_{MLE} = \frac{N}{\sum_{i=1}^{N} t_i} = \frac{1}{\overline{t}}$$

where $\overline{t} = \frac{1}{N}\sum_{t=1}^{N} t_i$.

By noting that $\ell''(\gamma) = -\frac{N}{\gamma^2}$, it follows that the information matrix is $\mathcal{I}(\gamma) = E(-\ell''(\gamma)) = \frac{N}{\gamma^2}$ and therefore, $\mathcal{I}^{-1}(\gamma) = \frac{\gamma^2}{N}$.

By property of MLE estimators, the asymptotic variance of the MLE estimator of γ is $(\mathcal{I}^{-1}(\gamma))$ equal to $\frac{\gamma^2}{N}$.

The mean of this distribution is E(t_i) $= \int_0^\infty t_i \gamma e^{-\gamma t_i} dt_i = \frac{1}{\gamma}\int_0^\infty x e^{-x} dx = \frac{1}{\gamma}$.

Hence, the MLE for the mean duration is $\overline{t}$ (by invariance property of MLE estimators; just take g(γ) $= \frac{1}{\gamma}$).

Again, by invariance property of the MLE estimators (take g(γ) $= \frac{1}{\gamma}$ *and* note that g'(γ) $= -\frac{1}{\gamma^2}$) the asymptotic variance of the mean duration is $(-\frac{1}{\gamma^2})^2 \frac{\gamma^2}{N} = \frac{1}{N\ \gamma^2}$. This follows from the $\delta-$ method.

The variance of unemplyment duration is Var(t$_i$) $= \int_0^\infty (\gamma - \frac{1}{\gamma})^2 e^{-\gamma t_i} dt_i = \frac{1}{\gamma^2}$, so we can see that the asymptotic variance of the mean duration is like the usual estimator of the variance of the sample mean.

Points to each part were assigned as follows (in the same order as in the question): 5, 4, 2, 3, 4 and 2 respectively.

Exercise 3:

Each estimator was worth 13 points. Comparing them was worth 1 point.

This exercise uses partition inverse formula. Since we first regress y on X, the residual vector we obtain from that regression is $e = My$ where $M = I - X\left(X'X\right)^{-1} X'$ is an idempotent and symmetric matrix.

Consider now the first model :

$$y = X\beta + z\delta + \varepsilon$$

where X is n x k, β is k x 1, z is n x 1, and δ is 1 x 1. Hence, estimation by OLS gives us :

$$\begin{bmatrix} \widehat{\beta} \\ \widehat{\delta} \end{bmatrix} = \begin{bmatrix} X'X & X'z \\ z'X & z'z \end{bmatrix}^{-1} \begin{bmatrix} X'y \\ z'y \end{bmatrix}$$

Using partitioned inverse formula, we have that

$$\begin{aligned} \widehat{\delta} &= -[z'z - z'X(X'X)^{-1}X'z]^{-1}z'X(X'X)^{-1}X'y + [z'z - z'X(X'X)^{-1}X'z]^{-1}z'y \\ &= -[z'\{I - X(X'X)^{-1}X'\}z]^{-1}z'X(X'X)^{-1}X'y + [z'\{I - X(X'X)^{-1}X'\}z]^{-1}z'y \\ &= -[z'Mz]^{-1}z'X(X'X)^{-1}X'y + [z'Mz]^{-1}z'y \\ &= [z'Mz]^{-1}z'[-X(X'X)^{-1}X' + I]y \\ &= (z'Mz)^{-1}z'My \end{aligned}$$

Consider now the second model :

$$e = z\delta \ + \ \tilde{\varepsilon}$$

where z is n x 1, and δ is 1 x 1. Hence, estimation by OLS gives us :

$$\begin{aligned} \delta^* &= (z'z)^{-1}z'e \\ &= (z'z)^{-1}z'My \end{aligned}$$

Consider now the last model :

$$e = X\beta + z\delta + \breve{\varepsilon}$$

where X is n x k, β is k x 1, z is n x 1, and δ is 1 x 1. Hence, estimation by OLS gives us :

$$\begin{bmatrix} \beta^+ \\ \delta^+ \end{bmatrix} = \begin{bmatrix} X'X & X'z \\ z'X & z'z \end{bmatrix}^{-1} \begin{bmatrix} X'e \\ z'e \end{bmatrix}$$

Using partitioned inverse formula, we have that

$$\begin{aligned}
\delta^{+} &= -[z'z - z'X(X'X)^{-1}X'z]^{-1}z'X(X'X)^{-1}X'e + [z'z - z'X(X'X)^{-1}X'z]^{-1}z'e \\
&= 0 + [z'\{I - X(X'X)^{-1}X'\}z]^{-1}z'e \qquad \text{since } X'e = 0 \\
&= [z'Mz]^{-1}z'e \\
&= (z'Mz)^{-1}z'My \qquad \text{since } \ e = My
\end{aligned}$$

Therefore, the conjecture is incorrect. It is true that $\widehat{\delta} = \delta^{+}$, but it is incorrect that $\widehat{\delta}$ *or* δ^{+} will be equal to δ^{*} (the three estimators will be equal if $z'z = (z'Mz)$ and this will happen if X and z are orthogonal).

Exercise 4:

(this part, 6 points)

We are told that one of the eigenvectors associated to V_n is 1_n. So, in order to calculate the corresponding eigenvalue, we write:

$$V1_n = \lambda 1_n$$

Hence,

$$\begin{aligned}
[I_n + \alpha 1_n 1_n']1_n &= \lambda 1_n \\
[I_n + \alpha 1_n 1_n']1_n - \lambda I_n 1_n &= 0 \\
[(1-\lambda)I_n + \alpha 1_n 1_n']1_n &= 0 \\
(1-\lambda)1_n + \alpha 1_n 1_n' 1_n &= 0 \\
(1-\lambda)1_n + \alpha n 1_n &= 0 \quad \text{since } 1_n' 1_n = n \\
[(1-\lambda) + \alpha n]1_n &= 0
\end{aligned}$$

This will give us the eigenvalue associated to the eigenvector 1_n associated to matrix V. So, we set $(1-\lambda) + \alpha n = 0$ and hence $\lambda = 1 + \alpha n$.

(This part is worth 10 points. Here it was expected that you show the procedure to get all the eigenvalues associated to V; the procedure alone is worth 6 points.)

We are told that one of the eigenvectors is 1_n, so a direct approach is to look at eigenvectors orthogonal to 1_n, that is, eigenvectors $y_1, ..., y_{n-1}$, such that $1_n' y_i = 0$ (for $i = 1, ..., n-1$). Then, the characteristic equations are ($i = 1, ..., n-1$)

$$\begin{aligned}
[I_n + \alpha 1_n 1_n']y_i &= \lambda_i y_i \\
I_n y_i + \alpha 1_n 1_n' y_i &= \lambda_i I_n y_i \\
I_n y_i &= \lambda_i I_n y_i \quad \text{since } 1_n' y_i = 0 \\
y_i &= \lambda_i y_i \\
&\Rightarrow \lambda_i = 1
\end{aligned}$$

Hence, the rest of the eigenvalues are all equal to one (so the "n" eigenvalues are $\lambda = 1$ with multiplicity n-1 and $\lambda = 1 + \alpha n$).

If you do not trust the above procedure, you can calculate the eigenvalues by the usual approach, but there is much more work to do. Here it is:

To calculate all the eigenvalues associated to V, we use 3 facts:

Fact 1: if you create a matrix B from a matrix A by susbtracting a multiple of one row of A to another row of A, the determinant is the same; that is, det A = det B.

Fact 2: if you create a matrix B from a matrix A by multiplying one row of matrix A by a constant r, then det B = r det A.

Fact 3: the determinant of a lower or upper triangular matrix is equal to the product of its diagonal elements.

For the proofs, you can see, for instance, Simon and Blume, Mathematics for Economist, 1994, page 729.

So, going back to the question, the charateristic equation is,

$$\begin{aligned}
Vx &= \lambda x \\
[I_n + \alpha 1_n 1_n']x &= \lambda x \\
[I_n + \alpha 1_n 1_n']x &= \lambda I_n x \\
[(1-\lambda)I_n + \alpha 1_n 1_n']x &= 0
\end{aligned}$$

Non trivial solution to this system requires that $\det[(1-\lambda)I_n + \alpha 1_n 1_n'] = 0$. Now, let's see how $[(1-\lambda)I_n + \alpha 1_n 1_n']$ looks like:

$$[(1-\lambda)I_n + \alpha 1_n 1_n'] = \begin{pmatrix}
1-\lambda+\alpha & \alpha & \alpha & . & . & . & \alpha & \alpha & \alpha \\
\alpha & 1-\lambda+\alpha & \alpha & \alpha & \alpha & . & . & . & \alpha \\
\alpha & \alpha & 1-\lambda+\alpha & \alpha & . & . & \alpha & . & \alpha \\
\alpha & \alpha & \alpha & . & \alpha & . & . & . & \alpha \\
\alpha & \alpha & \alpha & \alpha & . & \alpha & \alpha & \alpha & \alpha \\
\alpha & . & . & . & \alpha & . & \alpha & \alpha & \alpha \\
\alpha & . & \alpha & . & . & \alpha & 1-\lambda+\alpha & \alpha & \alpha \\
\alpha & . & . & . & . & . & \alpha & 1-\lambda+\alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & . & . & . & \alpha & 1-\lambda+\alpha
\end{pmatrix}$$

In order to calculate all the eigenvalues of this matrix, we will make it upper-triangular. To accomplish this, we do it in three steps. First, we are going to substract the last row from every row (we are using n-1 times fact 1), to get that:

$$\begin{aligned}
0 &= \det[(1-\lambda)I_n + \alpha 1_n 1_n'] \\
&= \det \begin{pmatrix}
1-\lambda+\alpha & \alpha & \alpha & . & . & . & \alpha & \alpha & \alpha \\
\alpha & 1-\lambda+\alpha & \alpha & \alpha & \alpha & . & . & . & \alpha \\
\alpha & \alpha & 1-\lambda+\alpha & \alpha & . & . & \alpha & . & \alpha \\
\alpha & \alpha & \alpha & . & \alpha & . & . & . & \alpha \\
\alpha & \alpha & \alpha & \alpha & . & \alpha & \alpha & \alpha & \alpha \\
\alpha & . & . & . & \alpha & . & \alpha & \alpha & \alpha \\
\alpha & . & \alpha & . & . & \alpha & 1-\lambda+\alpha & \alpha & \alpha \\
\alpha & . & . & . & . & . & \alpha & 1-\lambda+\alpha & \alpha \\
\alpha & \alpha & \alpha & \alpha & . & . & . & \alpha & 1-\lambda+\alpha
\end{pmatrix} \\
&= \det \begin{pmatrix}
1-\lambda & 0 & 0 & . & . & . & 0 & 0 & \lambda-1 \\
0 & 1-\lambda & 0 & 0 & 0 & . & . & . & \lambda-1 \\
0 & 0 & 1-\lambda & 0 & . & . & 0 & . & \lambda-1 \\
0 & 0 & 0 & 1-\lambda & 0 & . & . & . & \lambda-1 \\
0 & 0 & 0 & 0 & 1-\lambda & 0 & 0 & 0 & \lambda-1 \\
0 & . & . & . & 0 & 1-\lambda & 0 & 0 & \lambda-1 \\
0 & . & 0 & . & . & 0 & 1-\lambda & 0 & \lambda-1 \\
0 & . & . & . & . & . & 0 & 1-\lambda & \lambda-1 \\
\alpha & \alpha & \alpha & \alpha & . & . & . & \alpha & 1-\lambda+\alpha
\end{pmatrix}
\end{aligned}$$

Second, we are going to factor out $1-\lambda$ corresponding to each but the last row (the first up to the (n-1)th row). Here we are using n-1 times fact 2. Hence,

$$\begin{array}{rcl} 0 & = & \det[(1-\lambda)I_n + \alpha 1_n 1_n'] \\ & = & (1-\lambda)^{n-1} \det \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & . & . & . & -1 \\ 0 & 0 & 1 & 0 & . & . & 0 & . & -1 \\ 0 & 0 & 0 & 1 & 0 & . & . & . & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & . & . & . & 0 & 1 & 0 & 0 & -1 \\ 0 & . & 0 & . & . & 0 & 1 & 0 & -1 \\ 0 & . & . & . & . & . & 0 & 1 & -1 \\ \alpha & \alpha & \alpha & \alpha & . & . & . & \alpha & 1-\lambda+\alpha \end{pmatrix} \end{array}$$

Third, we are going to substract α times the first row from the last row; then we are going to substract α times the second row from the last row, and so on, up to substracting α times the (n-1)th row from the last row. Here again, we use n-1 times fact 1. So,

$$\begin{array}{rcl} 0 & = & \det[(1-\lambda)I_n + \alpha 1_n 1_n'] \\ & = & (1-\lambda)^{n-1} \det \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & . & . & . & -1 \\ 0 & 0 & 1 & 0 & . & . & 0 & . & -1 \\ 0 & 0 & 0 & 1 & 0 & . & . & . & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & . & . & . & 0 & 1 & 0 & 0 & -1 \\ 0 & . & 0 & . & . & 0 & 1 & 0 & -1 \\ 0 & . & . & . & . & . & 0 & 1 & -1 \\ \alpha & \alpha & \alpha & \alpha & . & . & . & \alpha & 1-\lambda+\alpha \end{pmatrix} \\ & = & (1-\lambda)^{n-1} \det \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & . & . & . & -1 \\ 0 & 0 & 1 & 0 & . & . & 0 & . & -1 \\ 0 & 0 & 0 & 1 & 0 & . & . & . & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & . & . & . & 0 & 1 & 0 & 0 & -1 \\ 0 & . & 0 & . & . & 0 & 1 & 0 & -1 \\ 0 & . & . & . & . & . & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1-\lambda+\alpha+(n-1)\alpha \end{pmatrix} \end{array}$$

We are now able to calculate all the eigenvalues. By fact 3, it follows that

$$\begin{array}{rcl} \det[(1-\lambda)I_n + \alpha 1_n 1_n'] & = & 0 \\ (1-\lambda)^{n-1}[1-\lambda+\alpha+(n-1)\alpha] & = & 0 \\ (1-\lambda)^{n-1}[1-\lambda+n\alpha] & = & 0 \end{array}$$

Therefore, the eigenvalues associated to the matrix V are $\lambda = 1$ (with multiplicity n-1) and $\lambda = 1+\alpha n$.

(last part, 4 points)

Finally, since V is a variance covariance matrix, it must be positive definite. The determinant of V is the product of its eigenvalues and hence (you can also calculated it using the above procedure) is $1+\alpha n$ and must be positive so that V is positive definite. In principle, for fixed n, α can be negative, but if we let n$\rightarrow\infty$, $\alpha < 0$ must be ruled out. However, any $\alpha \geq 0$ is fine, so in large economies there can be arbitrary positive correlation, but not much negative correlation (for fixed n). To see this, note that the correlation between ε_i and ε_j $(where\ \ i \neq j)$ is $\frac{\alpha}{1+\alpha}$ and $\frac{\alpha}{1+\alpha} \in [0,1)$ if $\alpha \geq 0$.