Econ 620

Three Likelihood Based Tests

Suppose that we have the log-likelihood function for n observations as;

$$L(\theta)$$

where $\theta \in \Theta \subset R^k$

The (unconstrained) MLE for θ is defined as

$$\widehat{\theta} = \arg\max_{\theta \in \Theta} L(\theta)$$

Moreover, we have the following hypotheses;

$$H_0; g(\theta) = 0 \qquad H_A; g(\theta) \neq 0$$

where $g(\cdot): R^k \to R^q$ and continuously differentiable with $\frac{\partial g}{\partial \theta}$ of full column rank. We define the constrained MLE for θ as

$$\overline{\theta} = \arg\max_{\theta \in \Theta} L(\theta) + g'(\theta)\lambda$$

where λ is a $(q \times 1)$ vector of Lagrangian multipliers.

We have three versions of tests to check the hypothesis;

$$W = ng\left(\widehat{\theta}\right)' \left[\frac{\partial g\left(\widehat{\theta}\right)}{\partial \theta'} i^{-1}\left(\widehat{\theta}\right) \frac{\partial g'\left(\widehat{\theta}\right)}{\partial \theta}\right]^{-1} g\left(\widehat{\theta}\right) \sim \chi^2(q)$$

$$LM = \frac{1}{n} s\left(\overline{\theta}\right)' i^{-1}\left(\overline{\theta}\right) s\left(\overline{\theta}\right) = \frac{1}{n}\overline{\lambda}' \left(\frac{\partial g\left(\overline{\theta}\right)}{\partial \theta'} i^{-1}\left(\overline{\theta}\right) \frac{\partial g'\left(\overline{\theta}\right)}{\partial \theta}\right) \overline{\lambda} \sim \chi^2(q)$$

$$LR = 2\left[L\left(\widehat{\theta}\right) - L\left(\overline{\theta}\right)\right] \sim \chi^2(q)$$

where $s\left(\overline{\theta}\right) = \frac{\partial L(\theta)}{\partial \theta} |_{\theta = \overline{\theta}}$ − score evaluated at $\theta = \overline{\theta}$, $\overline{\lambda}$ is the Lagrangian multiplier and $i(\theta)$ is the information matrix.

Three Tests in Normal Linear Model - Known Variance -

Suppose we have the following model;

$$y = X\beta + \varepsilon \qquad \varepsilon \sim N\left(0, \sigma^2 I\right)$$

where σ^2 is known. Without loss of generality, we can assume that $\sigma^2 = 1$. Furthermore, suppose that we have the following linear restrictions;

$$H_0; R\beta - r = 0 \qquad H_A; R\beta - r \neq 0$$

Then, the log likelihood function is given by

$$L(\beta) = -\frac{n}{2}\log 2\pi - \frac{1}{2}(y - X\beta)'(y - X\beta)$$

The first-order condition for maximization is;

$$\frac{\partial L(\beta)}{\partial \beta} = X'y - X'X\widehat{\beta} = 0$$

Hence, the unconstrained MLE is given by

$$\widehat{\beta} = (X'X)^{-1} X'y \tag{1}$$

On the other hand, we can obtain the constrained MLE;

$$\mathcal{L} = -\frac{n}{2}\log 2\pi - \frac{1}{2}(y - X\beta)'(y - X\beta) + (R\beta - r)'\lambda$$

The first-order conditions are;

$$\frac{\partial \mathcal{L}}{\partial \beta} = X'y - X'X\overline{\beta} + R'\overline{\lambda} = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = R\overline{\beta} - r = 0 \tag{3}$$

From (2),

$$\overline{\beta} = (X'X)^{-1} X'y + (X'X)^{-1} R'\overline{\lambda} = \widehat{\beta} + (X'X)^{-1} R'\overline{\lambda} \tag{4}$$

Multiplying (4) with R and rewriting gives;

$$\begin{aligned} \overline{\lambda} &= \left[R(X'X)^{-1} R'\right]^{-1}\left(R\overline{\beta} - R\widehat{\beta}\right) \\ &= \left[R(X'X)^{-1} R'\right]^{-1}\left(r - R\widehat{\beta}\right) \text{ from (3)} \\ &= -\left[R(X'X)^{-1} R'\right]^{-1}\left(R\widehat{\beta} - r\right) \end{aligned} \tag{5}$$

Substituting (5) back into (4) yields;

$$\overline{\beta} = \widehat{\beta} - (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1}\left(R\widehat{\beta} - r\right) \tag{6}$$

To get the information matrix, we differentiate the score with respect to β;

$$\frac{\partial^2 L(\beta)}{\partial\beta\partial\beta'} = -(X'X)$$

Therefore, the information matrix is given by;

$$\frac{1}{n}E\left(-\frac{\partial^2 L(\beta)}{\partial\beta\partial\beta'}\right) = i(\beta) = \left[\ \tfrac{1}{n}(X'X)\ \right] \tag{7}$$

Note that

$$g(\beta) = R\beta - r \Rightarrow g\left(\widehat{\beta}\right) = R\widehat{\beta} - r, \frac{\partial g\left(\widehat{\beta}\right)}{\partial\beta} = R$$

Then,

$$\begin{aligned} W &= n\left(R\widehat{\beta} - r\right)'\left[R\left[n(X'X)^{-1}\right]R'\right]^{-1}\left(R\widehat{\beta} - r\right) \\ &= \left(R\widehat{\beta} - r\right)'\left[R(X'X)^{-1} R'\right]^{-1}\left(R\widehat{\beta} - r\right) \end{aligned} \tag{8}$$

Again, note that

$$\frac{\partial g\left(\overline{\beta}\right)}{\partial\beta} = R,\ i\left(\overline{\beta}\right) = \frac{1}{n}(X'X)$$

remember that $\frac{\partial g(\beta)}{\partial\beta}$ and $i(\beta)$ do not depend on β in our case.

The LM statistic is given by;

$$\begin{aligned}
LM &= \frac{1}{n}\overline{\lambda}'\left(\frac{\partial g\left(\overline{\theta}\right)}{\partial\theta'}i^{-1}\left(\overline{\theta}\right)\frac{\partial g'\left(\overline{\theta}\right)}{\partial\theta}\right)\overline{\lambda} \\
&= \frac{1}{n}\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left[R\left(\frac{1}{n}X'X\right)^{-1}R'\right]\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \qquad (9)
\end{aligned}$$

the second equality comes from (5). From (7) and (8), we can check $W = LM$.

We have to evaluate the log-likelihood function at the constrained and the unconstrained MLE to find the LR statistic. Now,

$$\begin{aligned}
L\left(\widehat{\beta}\right) &= -\frac{n}{2}\log 2\pi - \frac{1}{2}\left(y-X\widehat{\beta}\right)'\left(y-X\widehat{\beta}\right) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\widehat{e}'\widehat{e} \\
L\left(\overline{\beta}\right) &= -\frac{n}{2}\log 2\pi - \frac{1}{2}\left(y-X\overline{\beta}\right)'\left(y-X\overline{\beta}\right) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\overline{e}'\overline{e}
\end{aligned}$$

Then,

$$LR = 2\left[L\left(\widehat{\beta}\right) - L\left(\overline{\beta}\right)\right] = \left[\overline{e}'\overline{e} - \widehat{e}'\widehat{e}\right] \qquad (10)$$

However,

$$\begin{aligned}
\overline{e}'\overline{e} &= \left(y-X\overline{\beta}\right)'\left(y-X\overline{\beta}\right) \\
&= \left[y - X\left(\widehat{\beta}-\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right)\right]' \\
&\left[y - X\left(\widehat{\beta}-\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right)\right] \text{ from(6)} \\
&= \left[\left(y-X\widehat{\beta}\right) + X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right]' \\
&\left[\left(y-X\widehat{\beta}\right) + X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right] \\
&= \left(y-X\widehat{\beta}\right)'\left(y-X\widehat{\beta}\right) + \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}R\left(X'X\right)^{-1}X'\left(y-X\widehat{\beta}\right) \\
&+ \left(y-X\widehat{\beta}\right)'X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&+ \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}R\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \left(y-X\widehat{\beta}\right)'\left(y-X\widehat{\beta}\right) + \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \widehat{e}'\widehat{e} + \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \qquad (11)
\end{aligned}$$

note that the second and the third term in the expansion vanish since $X'\left(y-X\widehat{\beta}\right) = X'\widehat{e} = 0$. From (10) and (11), we have

$$LR = \overline{e}'\overline{e} - \widehat{e}'\widehat{e} = \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)$$

Hence, we conclude that $LR = W = LM$. Moreover, we know that

$$LR = W = LM \sim \chi^2\left(q\right)$$

Since the model is based on the normally distributed error terms, we can actually obtain the exact distribution of the test statistic. Note that

$$\widehat{\beta} \sim N\left(\beta, \left(X'X\right)^{-1}\right)$$

remember that MLE is the least squares estimator in our case and we assumed that $\sigma^2 = 1$. Then,

$$R\widehat{\beta} \sim N\left(R\beta, R\left(X'X\right)^{-1}R'\right)$$
$$\left(R\widehat{\beta} - R\beta\right) \sim N\left(0, R\left(X'X\right)^{-1}R'\right)$$

Under the null hypothesis, $R\beta = r$.

$$\left(R\widehat{\beta} - r\right) \sim N\left(0, R\left(X'X\right)^{-1}R'\right)$$

Then, we can form a quadratic form of the normal variates to get a χ^2 random variable such that

$$\left(R\widehat{\beta} - r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta} - r\right) \sim \chi^2(q)$$

since $rank\left[R\left(X'X\right)^{-1}R'\right] = q$ by assumption. We don't need the $F-$test here since we know $\sigma^2 = 1$. The three likelihood based tests have the exact distribution as shown above. Since the distribution of error term is normal, we don't need the three tests based on asymptotic argument. However, we have shown that the asymptotic tests works even in the exact case.

Three Tests in Normal Linear Model - Unknown Variance -

Suppose we have the following model;

$$y = X\beta + \varepsilon \qquad \varepsilon \sim N\left(0, \sigma^2 I\right)$$

and the following linear restrictions;

$$H_0; R\beta - r = 0 \qquad H_A; R\beta - r \neq 0$$

Then, the log likelihood function is given by

$$L\left(\beta, \sigma^2\right) = L(\theta) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)$$

where $\theta = \left(\beta', \sigma^2\right)'$.

The first-order derivatives(scores) are;

$$\frac{\partial L(\theta)}{\partial\beta} = -\frac{1}{\sigma^2}\left[-X'y + X'X\beta\right] \qquad (12)$$
$$\frac{\partial L(\theta)}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(y - X\beta)'(y - X\beta) \qquad (13)$$

The second order derivatives are

$$\frac{\partial^2 L(\theta)}{\partial\beta\partial\beta'} = -\frac{1}{\sigma^2}X'X$$
$$\frac{\partial^2 L(\theta)}{\partial\beta\partial\sigma^2} = \frac{1}{\sigma^4}\left[-X'y + X'X\beta\right]$$
$$\frac{\partial^2 L(\theta)}{\partial\left(\sigma^2\right)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}(y - X\beta)'(y - X\beta)$$

Taking the expectations of the minus of the second derivatives;

$$
\begin{aligned}
E\left(-\frac{\partial^2 L(\theta)}{\partial\beta\partial\beta'}\right) &= \frac{1}{\sigma^2}X'X \\
E\left(-\frac{\partial^2 L(\theta)}{\partial\beta\partial\sigma^2}\right) &= \frac{1}{\sigma^4}\left[-X'E(y)+X'X\beta\right] \\
&= \frac{1}{\sigma^4}\left[-X'X\beta+X'X\beta\right]=0 \\
E\left(-\frac{\partial^2 L(\theta)}{\partial\left(\sigma^2\right)^2}\right) &= -\frac{n}{2\sigma^4}+\frac{1}{\sigma^6}E\left[(y-X\beta)'(y-X\beta)\right] \\
&= -\frac{n}{2\sigma^4}+\frac{1}{\sigma^6}E\left(\varepsilon'\varepsilon\right) = -\frac{n}{2\sigma^4}+\frac{1}{\sigma^6}n\sigma^2 \\
&= \frac{n}{2\sigma^4}
\end{aligned}
$$

Therefore, the information matrix is given by

$$
\frac{1}{n}E\left(-\frac{\partial^2 L(\theta)}{\partial\theta\partial\theta'}\right) = i\left(\beta,\sigma^2\right) = \begin{bmatrix} \frac{1}{n\sigma^2}X'X & 0 \\ 0 & \frac{1}{2\sigma^4}\end{bmatrix}
$$

The unrestricted MLE is given by;

$$
\begin{aligned}
\widehat{\beta} &= \left(X'X\right)^{-1}X'y \\
\widehat{\sigma}^2 &= \frac{1}{n}\left(y-X\widehat{\beta}\right)'\left(y-X\widehat{\beta}\right) = \frac{1}{n}\widehat{e}'\widehat{e}
\end{aligned}
$$

from (12) and (13).

We are now ready to form the Wald statistic.. Note that our null hypothesis is not involved in σ^2. In addition, since the information matrix is block diagonal between β and σ^2, we can ignore σ^2 in doing inferences on β. Identifying each elements in the statistic, we find that

$$
\begin{aligned}
W &= ng\left(\widehat{\beta}\right)'\left[\frac{\partial g\left(\widehat{\beta}\right)}{\partial\theta'}i^{-1}\left(\widehat{\beta}\right)\frac{\partial g'\left(\widehat{\beta}\right)}{\partial\theta}\right]^{-1}g\left(\widehat{\beta}\right) \\
&= n\left(R\widehat{\beta}-r\right)'\left[R\left(\frac{1}{n\widehat{\sigma}^2}X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \widehat{\sigma}^{-2}\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \qquad (14)
\end{aligned}
$$

note that the test statistic contains $\widehat{\sigma}^2$ unlike before. To find the LM statistic, we again form a constrained maximization problem;

$$
\mathcal{L} = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)+(R\beta-r)'\lambda
$$

The first-order conditions are;

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\beta} &= \frac{1}{\overline{\sigma}^2}\left[X'y-X'X\overline{\beta}\right]+R'\overline{\lambda}=0 \qquad (15) \\
\frac{\partial\mathcal{L}}{\partial\sigma^2} &= -\frac{n}{2\overline{\sigma}^2}+\frac{1}{2\overline{\sigma}^4}\left(y-X\overline{\beta}\right)'\left(y-X\overline{\beta}\right)=0 \qquad (16) \\
\frac{\partial\mathcal{L}}{\partial\lambda} &= R\overline{\beta}-r=0 \qquad (17)
\end{aligned}
$$

From (15),

$$
\overline{\beta} = \left(X'X\right)^{-1}X'y+\overline{\sigma}^2\left(X'X\right)^{-1}R'\overline{\lambda} = \widehat{\beta}+\overline{\sigma}^2\left(X'X\right)^{-1}R'\overline{\lambda} \qquad (18)
$$

Multiplying (18) with R and rewriting gives;

$$
\begin{aligned}
\overline{\lambda} &= \overline{\sigma}^{-2}\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\overline{\beta}-R\widehat{\beta}\right) \\
&= \overline{\sigma}^{-2}\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(r-R\widehat{\beta}\right) \text{ from (17)} \\
&= -\overline{\sigma}^{-2}\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \qquad (19)
\end{aligned}
$$

Substituting (19) back into (18) yields;

$$
\overline{\beta} = \widehat{\beta} - \left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \qquad (20)
$$

The LM statistic is now given by;

$$
\begin{aligned}
LM &= \frac{1}{n}\overline{\lambda}'\left(\frac{\partial g\left(\overline{\theta}\right)}{\partial\theta'}i^{-1}\left(\overline{\theta}\right)\frac{\partial g'\left(\overline{\theta}\right)}{\partial\theta}\right)\overline{\lambda} \\
&= \frac{1}{n}\left[\overline{\sigma}^{-2}\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right]'\left[R\left(\frac{1}{n\overline{\sigma}^{2}}X'X\right)^{-1}R'\right] \\
&\left[\overline{\sigma}^{-2}\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right] \\
&= \overline{\sigma}^{-2}\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left[R\left(X'X\right)^{-1}R'\right]\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \overline{\sigma}^{-2}\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \qquad (21)
\end{aligned}
$$

Note that (14) and (21) are different. We use $\widehat{\sigma}^2 = \frac{\widehat{e}'\widehat{e}}{n}$ in (14) and $\overline{\sigma}^2 = \frac{\overline{e}'\overline{e}}{n} = \frac{\left(y-X\overline{\beta}\right)'\left(y-X\overline{\beta}\right)}{n}$ in (21). The values of likelihood function at the unconstrained and the constrained MLE are given by;

$$
\begin{aligned}
L\left(\hat{\beta},\hat{\sigma}^2\right) &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2\widehat{\sigma}^2}\left(y-X\widehat{\beta}\right)'\left(y-X\widehat{\beta}\right) \\
&= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2\widehat{\sigma}^2}\widehat{e}'\widehat{e} = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\hat{\sigma}^2 - \frac{1}{2\widehat{\sigma}^2}n\widehat{\sigma}^2 \\
&= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\hat{\sigma}^2 - \frac{n}{2} \\
L\left(\bar{\beta},\bar{\sigma}^2\right) &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\bar{\sigma}^2 - \frac{1}{2\overline{\sigma}^2}\left(y-X\overline{\beta}\right)'\left(y-X\overline{\beta}\right) \\
&= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\bar{\sigma}^2 - \frac{1}{2\overline{\sigma}^2}\bar{e}'\bar{e} = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\bar{\sigma}^2 - \frac{1}{2\overline{\sigma}^2}n\bar{\sigma}^2 \\
&= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\bar{\sigma}^2 - \frac{n}{2}
\end{aligned}
$$

Then,

$$
\begin{aligned}
LR &= 2\left[L\left(\hat{\beta},\hat{\sigma}^2\right) - L\left(\bar{\beta},\bar{\sigma}^2\right)\right] \\
&= n\left[\log\bar{\sigma}^2 - \log\hat{\sigma}^2\right] = n\log\frac{\bar{\sigma}^2}{\hat{\sigma}^2} \qquad (22)
\end{aligned}
$$

A very expression can be obtained as;

$$\begin{aligned}
n\left(\bar{\sigma}^2-\hat{\sigma}^2\right) &= \bar{e}'\bar{e}-\hat{e}'\hat{e} = \left(y-X\overline{\beta}\right)'\left(y-X\overline{\beta}\right)-\left(y-X\widehat{\beta}\right)'\left(y-X\widehat{\beta}\right) \\
&= -2y'X\overline{\beta}+\overline{\beta}'X'X\overline{\beta}-2y'X\widehat{\beta}+\widehat{\beta}'X'X\widehat{\beta} \\
&= -2y'X\left[\widehat{\beta}-\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right] \\
&+\left[\widehat{\beta}-\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right]'X'X \\
&\left[\widehat{\beta}-\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\right] \\
&-2y'X\widehat{\beta}+\widehat{\beta}'X'X\widehat{\beta}\text{ from (20)} \\
&= -2y'X\widehat{\beta}+2y'X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&+\widehat{\beta}'X'X\widehat{\beta}-\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}R\left(X'X\right)^{-1}X'X\widehat{\beta} \\
&-\widehat{\beta}'X'X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&+\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}R\left(X'X\right)^{-1}X'X\left(X'X\right)^{-1} \\
&\times R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&-2y'X\widehat{\beta}+\widehat{\beta}'X'X\widehat{\beta} \\
&= 2y'X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)-\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}R\widehat{\beta} \\
&-\widehat{\beta}'R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)+\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= 2y'X\left(X'X\right)^{-1}R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)-2\widehat{\beta}'R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&+\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \left[2y'X\left(X'X\right)^{-1}-2\widehat{\beta}'\right]R'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&+\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \\
&= \left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)\ \text{ since }\widehat{\beta}'=y'X\left(X'X\right)^{-1}
\end{aligned}$$

In sum,

$$n\left(\bar{\sigma}^2-\hat{\sigma}^2\right)=\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right) \tag{23}$$

Now, from (14),(21) and (23);

$$W=\widehat{\sigma}^{-2}\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)=n\frac{\left(\bar{\sigma}^2-\hat{\sigma}^2\right)}{\widehat{\sigma}^2} \tag{24}$$

$$LM=\overline{\sigma}^{-2}\left(R\widehat{\beta}-r\right)'\left[R\left(X'X\right)^{-1}R'\right]^{-1}\left(R\widehat{\beta}-r\right)=n\frac{\left(\bar{\sigma}^2-\hat{\sigma}^2\right)}{\overline{\sigma}^2} \tag{25}$$

For completeness, we rewrite the LR statistic;

$$LR=n\log\frac{\bar{\sigma}^2}{\hat{\sigma}^2} \tag{26}$$

Recall the $F-statistic$;

$$F = \frac{\left(\bar{e}'\bar{e} - \widehat{e}'\widehat{e}\right)/q}{\widehat{e}'\widehat{e}/\left(n-k\right)} = \frac{\bar{e}'\bar{e} - \widehat{e}'\widehat{e}}{\widehat{e}'\widehat{e}}\frac{(n-k)}{q} = \frac{\frac{\bar{e}'\bar{e}}{n} - \frac{\widehat{e}'\widehat{e}}{n}}{\frac{\widehat{e}'\widehat{e}}{n}}\frac{(n-k)}{q}$$
$$= \frac{\bar{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2}\frac{(n-k)}{q} \tag{27}$$

From (24) and (27), we have;

$$F = \frac{1}{n}W\frac{(n-k)}{q} \Rightarrow W = n\frac{qF}{(n-k)} \tag{28}$$

From (25), (27), and (28), we have;

$$LM = \frac{\widehat{\sigma}^2}{\bar{\sigma}^2}W = \frac{\widehat{\sigma}^2}{\bar{\sigma}^2}\frac{qn}{(n-k)}F = \frac{n}{1+\frac{(n-k)}{qF}} \tag{29}$$

From (26) and (27), we have;

$$LR = n\log\left(\frac{\left(1+\frac{qF}{n-k}\right)\hat{\sigma}^2}{\hat{\sigma}^2}\right) = n\log\left(1+\frac{qF}{n-k}\right) \tag{30}$$

since $\bar{\sigma}^2 = \left(1+\frac{qF}{n-k}\right)\hat{\sigma}^2$ from (27). Let $\frac{qF}{(n-k)} = A$. Then,

$$W = nA, \qquad LR = n\log\left(1+A\right), \qquad LM = n\frac{A}{1+A}$$

Note that

$$\frac{A}{1+A} \leq \log\left(1+A\right) \leq A$$

We have

$$LM \leq LR \leq W \tag{31}$$

Moreover, as $n \to \infty$,

$$\begin{aligned} W &\to qF \qquad \text{from (28)} \\ LM &\to qF \qquad \text{from (29)} \\ LR &\to qF \end{aligned}$$

since

$$\lim_{n\to\infty} LR = \lim_{n\to\infty}\log\left(1+\frac{qF}{n-k}\right)^n = \log\lim_{n\to\infty}\left(1+\frac{qF}{n-k}\right)^n = \log\left[\exp\left(qF\right)\right] = qF$$