Economics 620, Lecture 8a: Asymptotics II

Uses of Asymptotic Distributions:

Suppose $\bar{X}_n - \mu \to 0$ in probability. (*What can be said about the distribution of $\bar{X}_n - \mu$?*)

In order to get distribution theory, we need to *norm* the random variable; we usually look at $n^{1/2}(\bar{X}_n - \mu)$.

Note that the random variable sequence $\{n^{1/2}\bar{X}_n - \mu)$, $n \geq 1\}$ does *not* converge in probability. (*Why not?*).

We might be able to make probability statements like

$$\lim_{n \to \infty} P(n^{1/2}(\bar{X}_n - \mu) < z) = F(z)$$

for some distribution F.

Then we could use F as an *approximate* distribution for $n^{1/2}(\bar{X}_n - \mu)$. This implies an *approximate* distribution for $\bar{X}_n$.

It is often easier to work with $Y_n = n^{1/2}(\bar{X}_n - \mu)/\sigma$.

Moment Generating Function

Definition: The moment generating function for the rv X (or the distribution f) is

$$m_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

The name comes from the fact that

$$\frac{d^r m}{dt^r} = \int_{-\infty}^{\infty} x^r e^{tx} f(x) dx = E(X^r)$$

when evaluated at $t = 0$

The subscript is dropped when unnecessary.

Moment Generating Function (cont'd):

Note the series expansion:

$$
\begin{aligned}
m(t) &= E(e^{tX}) = E(1 + Xt + \frac{1}{2!}(Xt)^2 + ...) \\
&= 1 + \alpha_1 t + \frac{1}{2}\alpha_2 t^2 + ...
\end{aligned}
$$

where $\alpha_r = EX^r$

(For example: $\alpha_1 = \mu$, $\alpha_2 = \mu^2 + \sigma^2$).

Property 1: The moment generating function of $\sum\limits_{i=1}^{n} X_i$ when X_i are independent is the product of the moment generating functions of X_i. (**Exercise**: *Prove this.*)

Property 2: Let X and Y be random variables with continuous densities $f(x)$ and $g(y)$. If the moment generating function of X is equal to that of Y in an interval $-h < t < h$, then $f = g$.

Moment Generating Function (cont'd)

Example: The moment generating function for $X \sim N(0,1)$ is

$$
\begin{aligned}
m(t) &= E(e^{tx}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}x^2} dx \\
&= e^{t^2/2} \left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx = ?
\end{aligned}
$$

Characteristic Function

The mgf does not always exist. The function $\phi_X(t) = Ee^{itX}$ always exists and is continuous and bounded.

We know $X_n \xrightarrow{d} X \Rightarrow \phi_{X_n}(t) \to \phi_X(t)$.

The converse

$$\phi_{X_n}(t) \to \phi_X(t) \ (\forall t) \Rightarrow X_n \xrightarrow{d} X$$

is also true

If $E|X| < \infty$ then $\phi'(0) = iEX$.

Similarly, if $EX^2 < \infty$ then $\phi''(0) = -EX^2$, etc.

CFs combine like mgfs.

Properties of c.f.s

$$\phi_X(t) = Ee^{itX} = E\cos tX + iE\sin tX$$

Thus $\phi(0) = 1, |\phi(t)| \leq 1$, and $\phi(t) = \overline{\phi(-t)}$.

where the bar indicates complex conjugate.

If X is symmetrically distributed, ϕ is real-valued.

Some cfs: Degenerate at $\mu, \phi(t) = e^{it\mu}$

Binomial(n,p), $\phi(t) = (pe^{it} + 1 - p)^n$

$N(\mu, \sigma^2), \phi(t) = e^{it\mu - \sigma^2 t^2/2}$

Using ϕ: Application 1, WLLN

Suppose $\{X_i\}$ are iid each with cf $\phi(t)$

Then,

$$Ee^{it\overline{X}} = \phi^n(t/n) = (1 + it\mu/n + o(1/n))^n$$

$$\rightarrow e^{it\mu}$$

The cf of the constant μ.

Application 2 will give a central limit theorem.

Using ϕ: Application 2, CLT

Central Limit Theorem: (*CLT*) (*Lindberg-Levy*) The distribution of $Y_n = n^{1/2}(\bar{X}_n - \mu)/\sigma$ as $n \to \infty$ is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

(standard normal)

Proof: Let $\phi_{X_i - \mu}(t)$ be the characteristic function of $(X_i - \mu)$. That is

$$\phi_{X_i - \mu}(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$$

where $o(t^2)$ is the remainder term such that $o(t^2)/t^2 \to 0$ as $t \to 0$.

CLT

We know that

$$Y_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}},$$

Hence

$$
\begin{aligned}
\phi_{Y_n}(t) &= \left[\phi_{X_i - \mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n \\
&= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \\
\Rightarrow \ln \phi_{Y_n}(t) &= n \ln\left[1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right] \\
&\approx n \ln\left[1 - \frac{t^2}{2n}\right]
\end{aligned}
$$

CLT

$$ln\phi_{Y_n}(t) \approx n \ln\left[1 - \frac{t^2}{2n}\right]$$

$$\Rightarrow \phi_{Y_n}(t) \rightarrow e^{-t^2/2}$$

as $n \rightarrow \infty$ (using $\ln(1 + x) \approx x$ for small x)

which is the cf of a standard normal random variable.

Point of the Central Limit Theorem: The distribution function of $\bar{X}_n$ for large n can be approximated by that of a normal with mean μ and variance σ^2/n.

Notes:

1.   Identical means and variances can be dropped straight-forwardly.   We need some restrictions on the variance sequence though.   In this case, we work with

$$Y_n = \frac{\sum_{i=1}^{n}(X_i - \mu_i)}{\left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2}}.$$

2.   Versions of the Central Limit Theorem with random vectors are also available. Just apply univariate theorems to all linear combinations.

3.   The basic requirement is that each term in the sum should make a negligible contribution.

Examples:

1. Estimation of mean μ from a sample of normal random variables: In this case, we estimate μ by $\bar{X}$, and the asymptotic approximation for the distribution of $\bar{X}$ or $(\bar{X} - \mu)$ is **exact**.

2. Consider $n^{1/2}(\hat{\beta} - \beta)$ where $\hat{\beta}$ is the LS estimator.

$$
\begin{aligned}
n^{1/2}(\hat{\beta} - \beta) &= n^{1/2}(X'X)^{-1}X'\varepsilon \\
&= [X'X/n]^{-1}n^{1/2}[X'\varepsilon/n]
\end{aligned}
$$

Where $[X'X/n]$ is the sample second moment matrix of the regressors.

$[X'X/n]$ is $O(1)$ or maybe $O_p(1)$ depending on assumptions.

Its lim or plim is Q, a KxK p.d. matrix.

Regression Example Cont'd

What about $n^{1/2}[X'\varepsilon/n] = \sqrt{n}(1/n)\sum x_i'\epsilon_i$?

This is $\sqrt{n}$ times a sample mean of $x_i'\epsilon_i$. These have $Ex_i'\epsilon_i = 0, Vx_i'\epsilon_i = \sigma^2 Q$ (discuss)

Under the assumption that regressors are well-behaved (i.e., contribution of any particular observation to $[X'\varepsilon/n]$ is negligible), we can apply a Central Limit Theorem and conclude that

$$n^{1/2}(\hat{\beta}-\beta) = [X'X/n]^{-1}n^{1/2}[X'\varepsilon/n] \xrightarrow{D} N(0,\sigma^2 Q^{-1}).$$

Consistent with previous results?

The Delta Method

The delta method is a "trick" for approximating the limiting distribution of a function of a statistic whose limiting distribution is known. From the CMT we know that $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$.

Suppose $X_n = \sqrt{n}(\overline{X}_n - \mu)$. What can we say about $\sqrt{n}(g(\overline{X}_n) - g(\mu))$?

Expand

$$g(\overline{X}_n) \approx g(\mu) + g'(\mu)(\overline{X}_n - \mu)$$

So

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)) \approx g'(\mu)\sqrt{n}(\overline{X}_n - \mu)$$

Hence if $\sqrt{n}(\overline{X}_n - \mu) \rightarrow N(0, \sigma^2)$

Then

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)) \rightarrow N(0, g'(\mu)^2 \sigma^2)$$

Multivariate CLT and Delta

Suppose $\{X_i\}$ are K-variate rv's , $EX = \mu, VX = \Sigma$

Then we consider $Y = t'X - t'\mu$, univariate with $EY = 0, VY = t'\Sigma t$ and apply our CLT to conclude

$$\sqrt{n}(\overline{X_n} - \mu) \rightarrow N(0, \Sigma)$$

For the Delta method, suppose $g : \mathbb{R}^K \rightarrow \mathbb{R}^m$ and suppose $\sqrt{n}(\overline{X}_n - \mu) \rightarrow N(0, \Sigma)$

Write $g(\overline{X}_n) \approx g(\mu) + g'(\mu)(\overline{X}_n - \mu)$ where $g'(\mu)$ is the mxk matrix of derivatives and conclude

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)) \rightarrow N(0, g'(\mu)\Sigma(g'(\mu))^T)$$