# Economics 620, Lecture 3:

# Simple Regression II

$\hat{\alpha}$ and $\hat{\beta}$ are the LS estimators

$\hat{y}_i = \hat{\alpha} + \hat{\beta} x_i$ are the estimated values

The Correlation Coefficient:

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}}.$$

$R^2 =$ (squared) correlation between $y$ and $\hat{y}$

Note: $\hat{y}$ is a linear function of $x$.

So $corr(y, \hat{y}) = |corr(y, x)|$.

# Correlation

*Proposition*: $-1 < r < 1$

$$r^2 = \frac{\left(\sum (x_i - \bar{x})(y_i - \bar{y})\right)^2}{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}.$$

Use Cauchy-Schwartz

$$\left(\sum x_i y_i\right)^2 \leq \sum x_i^2 \sum y_i^2$$

$$\Rightarrow r^2 \leq 1 \Rightarrow -1 \leq r \leq 1$$

*Proposition*: $\beta$ and $r$ have the same sign.

Proof:

$$\hat{\beta} = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2} = r \frac{\sqrt{\sum (y_i - \bar{y})^2}}{\sqrt{\sum (x_i - \bar{x})^2}}$$

# Correlation cont'd.

$$\sum e_i^2 = \sum(y_i - \bar{y})^2 - \hat{\beta}^2 \sum(x_i - \bar{x})^2$$

SSR $=$ TSS - SS explained by $x$

*Proposition*:

$$r^2 = 1 - \frac{SSR}{TSS} = 1 - \frac{\sum e_i^2}{\sum(y_i - \bar{y})^2}$$

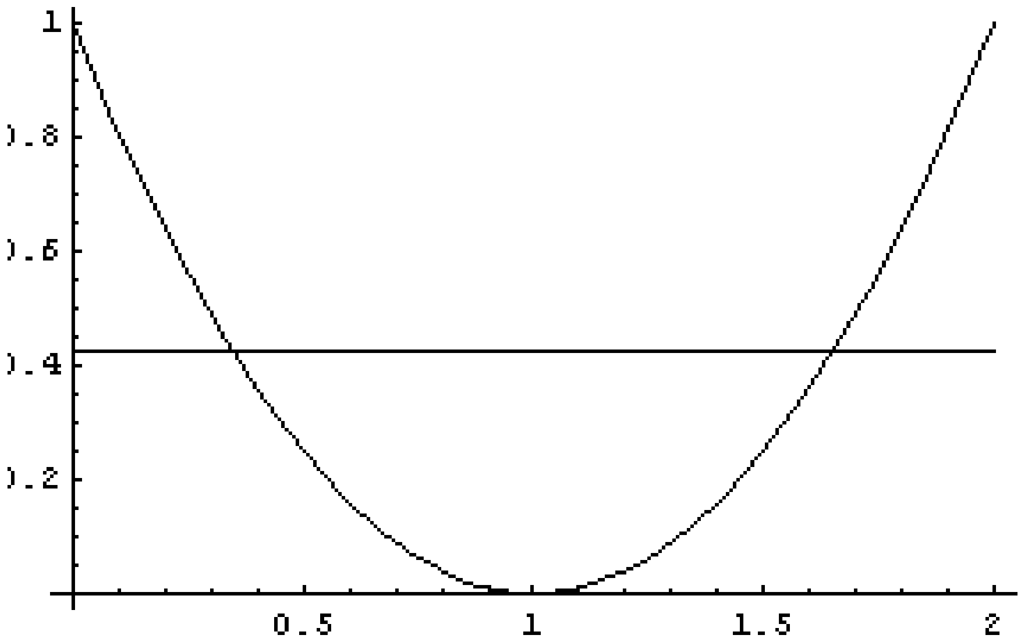
Proof:

$$\frac{\sum e_i^2}{\sum(y_i - \bar{y})^2} = 1 - \hat{\beta}^2 \frac{\sum(x_i - \bar{x})^2}{\sum(y_i - \bar{y})^2} = 1 - r^2$$

$$\Rightarrow r^2 = 1 - \frac{\sum e_i^2}{\sum(y_i - \bar{y})^2}$$

# Warning: Correlation $\neq$ Dependence




Variables are completely dependent, correlation is zero. Correlation is a measure of linear dependence.

## The Likelihood Function

A complete specification of the model

Conditional distribution of observables

Conditional on regressors x "exogenous variables" - variables determined outside the model

Conditional on parameters $P(y|x, \alpha, \beta, \sigma^2)$

Previously, specified only mean and maybe variance

Incompletely specified = "semiparametric"

Point estimate: MLE – intuition

Details, asy. justification lecture 9.

# Maximum Likelihood Estimators

Assumptions: Normality

$$
\begin{aligned}
p(y|x) &= N(\alpha + \beta x, \sigma^2) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{y - \alpha - \beta x}{\sigma}\right)^2\right)
\end{aligned}
$$

Likelihood Function:

$$
\begin{aligned}
L(\alpha, \beta, \sigma^2) &= \textstyle\prod_{i=1}^{n}(p(y_i|x_i) \\
= (2\pi\sigma^2)^{(-n/2)} & \exp\left(\frac{-1}{2\sigma^2} \textstyle\sum_{i=1}^{n}(y_i - \alpha - \beta x_i)^2\right)
\end{aligned}
$$

The maximum likelihood (ML) estimators maximize $L$. The log likelihood function is

$$
\ell(\alpha, \beta, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\textstyle\sum_{i=1}^{n}(y_i - \alpha - \beta x_i)^2
$$

# Maximum Likelihood cont'd.

*Proposition*: The LS estimators are also the ML estimators. What is the maximum in $\sigma^2$?

$$\sigma^2_{ML} = \sum_{i=1}^{n}(y_i - \hat{\alpha} - \hat{\beta}x_i)^2/n$$

Why?

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i - \alpha - \beta x_i)^2$$

$$\Rightarrow \sigma^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{\alpha} - \hat{\beta}x_i)^2$$

is this a maximum in $\sigma$?

$$\frac{\partial^2 \ell}{\partial(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum(y_i - \alpha - \beta x_i)^2 = \frac{-n}{2\sigma^4} < 0$$

# Distribution of Estimators

These are linear combinations of normal random variables, hence they are **normal**. The means and variances have already been obtained:

*Distribution of* $s$ *and* $\sigma$

*Fact*: $\sum e^2$ can be written as a sum of squares of $(n-2)$ independent normal random variables with means zero and variances $\sigma^2$.

*Proposition*: $s^2$ is unbiased and $Vs^2 = 2\sigma^4/(n-2)$.

*Proof*: Note that $(n-2)s^2/\sigma^2$ is distributed as $\chi^2(n-2)$

## More Distributions

$$\Rightarrow E(s^2/\sigma^2)(n-2) = (n-2) \Rightarrow E(s^2) = \sigma^2$$

$$\Rightarrow V(s^2/\sigma^2)(n-2) = 2(n-2)$$
$$\text{so } V(s^2) = 2\sigma^4/(n-2)$$

*Proposition*: $s^2$ has higher variance than $\sigma^2_{ML}$

*Proof*: Note that $\frac{n\sigma^2_{ML}}{\sigma^2}$ is distributed as

$$\chi^2(n-2)$$

$$\Rightarrow E\sigma^2_{ML} = \frac{\sigma^2(n-2)}{n}$$

$$\Rightarrow V\left(\frac{n\sigma^2_{ML}}{\sigma^2}\right) = 2(n-2) \Rightarrow V(\sigma^2_{ML}) = \frac{2\sigma^4(n-2)}{n^2}$$

$$\Rightarrow \frac{V(s^2)}{V(\sigma^2_{ML})} = \frac{1/(n-2)}{(n-2)n^2} = \frac{n^2}{(n-2)^2} > 1$$

# Inference

$\hat{\beta} \sim N(\beta, \sigma_\beta^2)$ where $\sigma_\beta^2 = \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \Rightarrow \frac{\hat{\beta}-\beta}{\sigma_\beta} \sim n(0,1)$

*Definition*: A 95% confidence interval for $\hat{\beta}$ is given by $(\hat{\beta} \mp z^*_{0.025}\sigma_\beta)$ where $z$ is standard normal.

Problem: The variance is unknown.

*Fact*: If $z \sim n(0,1)$ and $v \sim \chi^2(k)$ **and** they are independent, then $t = \frac{z}{\sqrt{v/k}}$ is distributed as $t(k)$.

*Proposition*:

$$\frac{\hat{\beta}-\beta}{s/\sqrt{\sum(x_i-\bar{x})^2}} \sim t(n-2)$$

# Proof:

$$\frac{(\hat{\beta}-\beta)\sqrt{\sum(x_i-\bar{x})^2}}{\sigma} \sim n(0,1)$$

$$\frac{s^2}{\sigma^2}(n-2) \sim \chi^2(n-2)$$

$$\frac{\frac{(\hat{\beta}-\beta)\sqrt{\sum(x_i-\bar{x})^2}}{\sigma}}{s/\sigma} = \frac{(\hat{\beta}-\beta)}{s/\sqrt{\sum(x_i-\bar{x})^2}} \sim t(n-2)$$

Independence?

$$\begin{aligned}
E(\hat{\beta}-\beta)e_j &= E[(\hat{\beta}-\beta)(e_j-\bar{e})] \\
&= E[(\hat{\beta}-\beta)((\alpha-\hat{\alpha})+(\beta-\hat{\beta})x_j+\varepsilon_j \\
&\quad -(\alpha-\hat{\alpha})-(\beta-\hat{\beta})\bar{x}-\bar{\varepsilon})] \\
&= [(\hat{\beta}-\beta)(-(\hat{\beta}-\beta)(x_j-\bar{x})+(\varepsilon_j-\bar{\varepsilon}))] \\
&= -(x_j-\bar{x})E[(\hat{\beta}-\beta)^2] \\
&\quad +E[(\hat{\beta}-\beta)(\varepsilon_j-\bar{\varepsilon})] \\
&= \frac{-\sigma^2(x_j-\bar{x})}{\sum(x_i-\bar{x})^2} + E\frac{(\varepsilon_j-\bar{\varepsilon})\sum(x_i-\bar{x})\varepsilon_i}{\sum(x_i-\bar{x})^2}
\end{aligned}$$

# Continuation of independence argument

$$E\frac{(\varepsilon_j - \bar{\varepsilon})\sum(x_i - \bar{x})\varepsilon_i}{\sum(x_i - \bar{x})^2} = \frac{\sigma^2(x_j - \bar{x})}{\sum(x_i - \bar{x})^2} - E\frac{\bar{\varepsilon}\sum(x_i - \bar{x})\varepsilon_i}{\sum(x_i - \bar{x})^2}.$$

$$E\frac{\bar{\varepsilon}\sum(x_i - \bar{x})\varepsilon_i}{\sum(x_i - \bar{x})^2} = 0.$$

Thus,

$$E(\hat{\beta} - \beta)e_j = 0.$$

## Violations of Assumptions

I. $Ey_i = \alpha + x_i\beta$

II. $V(y_i|x_i) = V(\varepsilon_i) = \sigma^2$

The alternative is $\sigma_i^2$ different across observations (*heteroskedasticity*).

Is the LS estimator unbiased? Is it BLUE?

If the $\sigma_i$ are known we can run the 'transformed' regression, and will get best linear unbiased estimates and correct standard errors.

$w_i = 1/\sigma_i$, let $w_i y_i = \alpha w_i + \beta x_i w_i + \varepsilon_i w_i$.

$Ew_i y_i = \alpha w_i + \beta x_i w_i$ and $V(w_i y_i) = V(\varepsilon_i w_i) = 1$

The Gauss-Markov Theorem tells that LS is BLUE in the transformed model.

## Heteroskedasticity continued

The LS estimator in the transformed model is

$$\hat{\beta}_w = \frac{\sum(x_i w_i - \overline{xw}) w_i y_i}{\sum(x_i w_i - \overline{xw})^2} \neq \hat{\beta}$$

with

$$V(\hat{\beta}) = \frac{\sum(x_i - \bar{x})^2 \sigma_i^2}{\left(\sum(x_i - \bar{x})^2\right)^2}$$

Note: The variance of $\beta_w$ is less than the variance of $\beta$.

"Heteroskedasticity Consistent" standard errors:

$$V(\hat{\beta}) = E\left[\frac{\sum(x_i - \bar{x})\varepsilon_i}{\sum(x_i - \bar{x})^2}\right]^2 = E\left[\frac{\sum(x_i - \bar{x})^2 \varepsilon_i^2}{\left(\sum(x_i - \bar{x})^2\right)^2}\right]$$

insert $e$ for $\varepsilon$ and remove the expectation.

## More on Heteroskedasticity

Essentially this works because $\sum \hat{e}_i^2/n$ is a reasonable estimator for $\sum \sigma_i^2/n$, although of course, $\hat{e}_i^2$ is not a good estimator for $\sigma_i^2$.

*Testing for heteroskedasticity*:

Split the sample; regress $e^2$ on stuff

III. $E\varepsilon_i\varepsilon_j = 0$

The alternative is $E\varepsilon_i\varepsilon_j \neq 0$

Is the LS estimator unbiased? Is it BLUE?

*Testing for correlated errors*:

We need a hypothesis about the correlation.

# More (last) on violations of assumptions

IV. Normality

$E(y_i|x_i) = \alpha + \beta x_i$; $V(y_i|x_i) = \sigma^2$ but $\varepsilon_i \sim f(\varepsilon) \neq N(0, \sigma^2)$

The usual suspect is a heavy-tailed distribution. Is the LS estimator unbiased? Is it BLUE?

*Example*:

$$f(\varepsilon) = \frac{1}{2\phi} \exp\left(-\left|\varepsilon/\phi\right|\right)$$

The variance of the ML estimator is half that of the LS estimator asymptotically. The minimum absolute deviation (MAD) estimator works. It is a **robust** estimator.