

## Lecture 15: Introduction to the Simultaneous Equations Model

The statistics framework for the simultaneous equations model (SEM) is the multivariate regression. For example, let  $E(y|X, \Pi) = \Pi X$  and  $V(y|X, \Pi) = \Sigma$  where

$$\begin{array}{ll} y: 2 \times 1 & X: 3 \times 1 \\ \Pi: 2 \times 3 & \Sigma: 2 \times 2. \end{array}$$

There are really 5 variables  $(y, X)$  but we *condition* on  $X$ . This is a modelling choice.

Suppose  $y = (q, p)$ , a quantity and a price, and  $X = (x_1, x_2, x_3)$  where  $x_3 = 1$  with

$$\left. \begin{aligned} Eq &= \pi_{11}x_1 + \pi_{12}x_2 + \pi_{13} \text{ and} \\ Ep &= \pi_{21}x_1 + \pi_{22}x_2 + \pi_{23}. \end{aligned} \right\} (I)$$

Perhaps,  $x_1$  is “weather” and  $x_2$  is income and we are modeling an agricultural market.

Economically,

$$\left. \begin{aligned} q^S &= \beta_{11}p + \gamma_{11}x_1 + \gamma_{13} \\ q^D &= \beta_{21}p + \gamma_{22}x_2 + \gamma_{23} \\ q^S &= q^D. \end{aligned} \right\} (II)$$

This is more natural - a demand and supply equation and an equilibrium condition determining  $q$  and  $p$ .

*Definition:* Equation system (II) is the structural form.

*Definition:* Multivariate regression (I) is the reduced form.

Note: Restrictions on structure like  $\gamma_{12} = 0$ , impose nonlinear restrictions on the reduced form.

The reduced form model achieves the “reduction in data dimension” possible through statistical modeling. Thus, a reduced form parametrization is in 1–1 correspondence with the sufficient statistic.

Additional reduction in dimension is possible with the use of theoretical models; these lead to structural models (or structural parametrizations).

## Indirect Least Squares:

Solving the structure for  $p$  and  $q$  yield

$$q = \frac{\gamma_{11}\beta_{21}}{\beta_{11} - \beta_{21}}x_1 - \frac{\beta_{11}\gamma_{22}}{\beta_{11} - \beta_{21}}x_2 + \frac{\beta_{11}\gamma_{23} - \beta_{21}\gamma_{13}}{\beta_{11} - \beta_{21}}$$

$$p = \frac{\gamma_{11}}{\beta_{11} - \beta_{21}}x_1 - \frac{\gamma_{22}}{\beta_{11} - \beta_{21}}x_2 + \frac{\gamma_{23} - \gamma_{13}}{\beta_{11} - \beta_{21}}.$$

These equations determine  $\pi$  in terms of  $\beta$  and  $\gamma$ .

Given

$$\begin{aligned}\beta_{11} &= -\frac{\pi_{12}}{\pi_{22}}; \quad \beta_{21} = -\frac{\pi_{11}}{\pi_{21}}; \\ \gamma_{11} &= -\pi_{21}(\beta_{11} - \beta_{21}), \text{ etc.}\end{aligned}$$

There are 6 elements of  $\pi$  and 6  $\beta$  and  $\gamma$ . Thus,  $\Pi$  can be estimated by the LS method and we can solve for  $\beta$  and  $\gamma$ . This is the Indirect Least Squares method.

Why not estimate  $\beta_{11}$ ,  $\gamma_{11}$  and  $\gamma_{13}$  by regressing  $q$  on  $p$  and  $x_1$ ?

## Identification:

### *Underidentification:*

Suppose  $\gamma_{22} = 0$ . Then

$$q = \pi_{11}x_1 + \pi_{13}$$

$$p = \pi_{21}x_1 + \pi_{23}$$

is the implied reduced form. Then the 4 reduced form parameters determine  $\beta_{12}$  and  $\gamma_{23}$  (the demand curve) but no more. This supply curve is underidentified.

### *Overidentification:*

Suppose we still have  $\gamma_{22} = 0$  and  $\gamma_{12} \neq 0$ . Then

$$\begin{aligned} q^S &= \beta_{11}p + \gamma_{11}x_1 + \gamma_{12}x_2 + \gamma_{13} \\ q^D &= \beta_{21}p + \gamma_{21}x_1 + \gamma_{22}x_2 + \gamma_{23} \\ q^S &= q^D. \end{aligned}$$

In reduced form,

$$\begin{aligned} q &= \frac{\gamma_{11}\beta_{21}x_1}{\beta_{11} - \beta_{21}} + \frac{\gamma_{12}\beta_{21}x_2}{\beta_{11} - \beta_{21}} + \text{constant} \\ p &= \frac{\gamma_{11}x_1}{\beta_{11} - \beta_{21}} + \frac{\gamma_{12}x_2}{\beta_{11} - \beta_{12}} + \text{constant}. \end{aligned}$$

Now  $\beta_{21} = \pi_{11}/\pi_{21} = \pi_{12}/\pi_{22}$ .

If  $\pi$  is estimated, the 2 estimates of  $\beta_{21}$  will not usually be equal. In this case,  $\beta_{21}$  is overidentified.

It is natural to estimate  $\beta_{21}$  by a weighted average. The whole point of SEM estimation is finding the right weights to use to combine the reduced form estimates.

*Underidentification as (exact) collinearity:*

$$q^S = \beta_{11}p + \gamma_{11}x_1 + \gamma_{13} \text{ (underidentification)}$$

$$q^D = \beta_{21}p + \gamma_{23} \text{ (identification)}$$

$$q^S = q^D$$



The weighted average of  $q^S$  and  $q^D$  is

$$q = (\lambda\beta_{11} + (1 - \lambda)\beta_{21})p + \lambda\gamma_{11}x_1$$

$$+ \lambda\gamma_{13} + (1 - \lambda)\gamma_{23}.$$

This is indistinguishable from  $q^S$  when parameters are unknown.

The point is that if we fit an equation like  $q^S$ , we can't say what the parameters are other than a mix of demand and supply parameters.

## Classical identification conditions:

First, restrict attention to a 2-equation system.

Make a table of which variables are in which equations. For example, the following table does this for the original model:

	$q$	$p$	$x_1$	$x_2$	constant
$q^S$	✓	✓	✓		✓
$q^D$	✓	✓		✓	✓

*Order condition:* There is at least one blank space in the row of the identified equation. This is a necessary condition.

*Rank condition:* The variable left out of the equation considered must appear in the other. This is a necessary condition.

The order condition and the rank condition together are both necessary and sufficient.

Note that we are considering only the class of exclusion restrictions on the structure. Rank and order condition are not necessary when broader restrictions are permitted.

Consider the identification of the first equation in the following  $G$ -equation table. Note that  $y$  is  $G \times 1$  and  $X$  is  $K \times 1$ .

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	$y_1$	$y_2$	...	...	$y_G$	$x_1$	$x_2$	...	...	$x_K$
Eq.1	✓	✓			✓		✓			
Eq.2										
...										
...										
Eq. G										
						$A_1$				$A_2$

*Order condition:* If the number of blank spaces in row 1 is greater than or equal to the number of endogenous variables minus one (i.e.,  $G - 1$ ), then equation 1 is identified. (*Why?*)

*Rank condition:* The matrix formed by taking coefficients corresponding to blanks in row 1, that is, the matrix

$$A = [A_1 : A_2 : ?],$$

has rank  $G - 1$ .

*Other sources of identification:*

Consider the following structural model:

$$q^S = \beta_{11}p + \gamma_{11}x_1 + \gamma_{13} + u$$

$$q^D = \beta_{21}p \quad \quad \quad + \gamma_{23} + v$$

$$q^S = q^D.$$

Assume that  $Euv = 0$ . Note that this is a restriction. Assume further that  $Eu = Ev = 0$ ,  $Eu^2 = \sigma_u^2$  and  $Ev = \sigma_v^2$ .

Consider the following equation:

$$q = (\lambda\beta_{11} + (1 - \lambda)\beta_{21})p + \lambda\gamma_{11}x_1 + \text{constant} + \lambda u + (1 - \lambda)v$$

$$q = \beta_{11}p + \gamma_{11}x_1 + \text{constant} + u^*.$$

This equation is distinguishable from  $q^S$  since  $u^*$  is correlated with  $v$  for  $\lambda \neq 1$ .

In reduced form:

$$q = \pi_{11}x_1 + \pi_{12} + W_1$$

$$p = \pi_{21}x_1 + \pi_{22} + W_2$$

$$W_1 = (\beta_{21}u - \beta_{11}v)/(\beta_{21} - \beta_{11})$$

$$W_2 = (u - v)/(\beta_{21} - \beta_{11}).$$

Now  $\pi$  can be estimated by the LS method and so can

$$EW_1^2 = \sigma_{11} = (\beta_{21}^2 \sigma_u^2 + \beta_{11}^2 \sigma_v^2) / (\beta_{21} - \beta_{11})^2$$

$$EW_2^2 = \sigma_{22} = (\sigma_u^2 + \sigma_v^2) / (\beta_{21} - \beta_{11})^2$$

$$EW_1 W_2 = \sigma_{12} = (\beta_{21} \sigma_u^2 + \beta_{11} \sigma_v^2) / (\beta_{21} - \beta_{11})^2.$$

These equations and our earlier formulas can be used to solve for all the structural parameters.

Note that the rank and order conditions are not satisfied, illustrating that the conditions are not necessary once the set of allowable restrictions is expanded.