Economics 620, Lecture 16: Estimation of Simultaneous Equations Models

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Consider $y_1 = Y_2\gamma + X_1\beta + \varepsilon_1$ which is an equation from a system.

We can rewrite this at $y_1 = Z\delta + \varepsilon_1$ where $Z = [Y_2 \ X_1]$ and $\delta = [\gamma' \ \beta']'$.

Note that Y_2 is jointly determined with y_1, so

$$plim(1/N)Z'\varepsilon_1 \neq 0 \ (usually).$$

IV Estimation:

The point of IV estimation is to find a matrix of <u>instruments</u> W so that

$$\text{plim}\frac{W'\varepsilon_1}{N} = 0$$

and

$$\text{plim}\frac{W'Z}{N} = Q$$

where Q is nonsingular.

The IV estimator is $(W'Z)^{-1}W'y_1$. As in the lecture on dynamic models, multiplying the model by the transpose of the matrix of instruments yields $W'y_1 = W'Z\delta + W'\varepsilon_1$ which gives $\hat{\delta}_{IV}$.

Asymptotic distribution of $\hat{\delta}_{IV}$:

Note that $\hat{\delta}_{IV} - \delta = (W'Z)^{-1}W'\varepsilon_1$. Assume that

$$\frac{W'\varepsilon_1}{\sqrt{N}} \to N\left(0, \sigma^2 \frac{W'W}{N}\right).$$

(*Is this a sensible assumption*? *Recall the CLT.*)

Then

$$\sqrt{N}(\hat{\delta}_{IV} - \delta) \to N(0, \sigma^2 \textstyle\sum_{\delta})$$

where

$$\sum_{\delta} = N(W'Z)^{-1}W'W(W'Z)^{-1} = (1/N)Q^{-1}W'WQ^{-1}.$$

The question is what to use for W. Suppose we use X.

Multiplying by the tranpose of the matrix of instruments gives $X'y_1 = X'Z\delta + X'\varepsilon_1$.

For this system of equations to have a solution, $X'Z$ has to be square and nonsingular. When is this possible?

Note the following dimensions: X is $N \times K$, X_1 is $N \times K_1$ and Y_2 is $N \times (G_1 - 1)$. This, of course, requires $K = K_1 + G_1 - 1$.
(*Recall the* order condition: $K \geq K_1 + G_1 - 1$.)

Thus, the above procedure works when the equation is just identified.

The resulting IV estimates are indirect least squares which we saw last time.

Suppose $K < K_1 + G_1 - 1$. Then what happens? Consider the supply and demand example. This is the underidentified case.

Suppose $K > K_1 + G_1 - 1$. Then $X'y_1 = X'Z\delta + X'\varepsilon_1$ is K equations in $K_1 + G_1 - 1$ unknowns (setting $X'\varepsilon_1$ to 0 which is its expectation). We could choose $K_1 + G_1 - 1$ equations to solve for δ- there are many ways to do this, typically leading to different estimates. This is the overidentified case.

Another way to look at this case is as a regression model - with K "observations" on the dependent variable.

We could apply the LS method, but the GLS is more efficient since $V(X'\varepsilon_1) = \sigma^2(X'X)(\neq \sigma^2 I)$.

The observation matrix is $X'y_1$ and $X'Z$. GLS gives the estimator

$$\hat{\delta} = [Z'X(X'X)^{-1}X'Z]^{-1}Z'X(X'X)^{-1}X'y_1.$$

In the just-identified case (where $X'Z$ is invertible),

$$\begin{aligned} \hat{\delta} &= (X'Z)^{-1}X'X(Z'X)^{-1}Z'X(X'X)^{-1}X'y_1 \\ &= (X'Z)^{-1}X'y_1 = \hat{\delta}_{IV} \text{ with } W = X. \end{aligned}$$

TWO-STAGE LEAST SQUARES:

Return to the overidentified case:

$$\hat{\delta} = [Z'X(X'X)^{-1}X'Z]^{-1}Z'X(X'X)^{-1}X'y_1.$$

Proposition: The estimator

$$\hat{\delta} = [Z'X(X'X)^{-1}X'Z]^{-1}Z'X(X'X)^{-1}X'y_1$$

is the two-stage least squares (2SLS or TSLS) estimator.

Why is $\hat{\delta}$ called the TSLS estimator?

Let $\bar{M} = X(X'X)^{-1}X' = I - M$.
Then $\hat{\delta} = (Z'\bar{M}Z)^{-1}Z'\bar{M}y_1$.

We will write out the expression for $\hat{\delta}$.

$$\hat{\delta} = \left[\begin{array}{cc} \hat{Y}_2'\hat{Y}_2 & \hat{Y}_2'X_1 \\ X_1'\hat{Y}_2 & X_1'X_1 \end{array} \right]^{-1} \left[\begin{array}{c} \hat{Y}_2'y_1 \\ X_1'y_1 \end{array} \right].$$

Now: $\bar{M}Y_2 = X(X'X)^{-1}X'Y_2 = \hat{Y}_2 = X\hat{\Pi}_2$ which is the LS predictor of Y_2.

$$Z'\bar{M}Z = \left[\begin{array}{cc} Y_2'\bar{M}Y_2 & Y_2'\bar{M}X_1 \\ X_1'\bar{M}Y_2 & X_1'\bar{M}X_1 \end{array} \right]$$

Note that $X_1'\bar{M}X_1 = X_1'X_1.\big(R[X_1] \subset R[X] \Rightarrow \bar{M}X_1 = X_1; \bar{M}X = X\big).$

Also: $Y_2'\bar{M}Y_2 = Y_2'\bar{M}\bar{M}Y_2 = \hat{Y}_2'\hat{Y}_2.$

So,

$$\hat{\delta} = \left[\begin{array}{cc} \hat{Y}_2'\hat{Y}_2 & \hat{Y}_2'X_1 \\ X_1'\hat{Y}_2 & X_1'X_1 \end{array} \right]^{-1} \left[\begin{array}{c} \hat{Y}_2'y_1 \\ X_1'y_1 \end{array} \right].$$

$\hat{\delta}$ is the coefficient vector from a regression of y_1 on $\hat{Y}_2$ and X_1.

Interpretation as 2SLS? Interpretation as IV?

Proposition: 2SLS is IV estimation with $W = [\hat{Y}_2 X_1]$.

Proof: Note that

$$W'Z = \begin{bmatrix} \hat{Y}_2' Y_2 & \hat{Y}_2' X_1 \\ X_1' \hat{Y}_2 & X_1' X_1 \end{bmatrix} = \begin{bmatrix} \hat{Y}_2' \hat{Y}_2 & \hat{Y}_2' X_1 \\ X_1' \hat{Y}_2 & X_1' X_1 \end{bmatrix}.$$

This is the matrix appearing inverted in $\hat{\delta}$. ■

Asymptotic distribution of $\hat{\delta}$: We know this from IV results.

Note that $\hat{\delta} = \delta + (Z'\bar{M}Z)^{-1}Z'\bar{M}\varepsilon_1$. The asymptotic variance of $N^{1/2}(\hat{\delta} - \delta)$ is the asymptotic variance of $N^{1/2}(Z'\bar{M}Z)^{-1}Z'\bar{M}\varepsilon_1 = u$.

$$\begin{aligned} Var(u) &= N\sigma^2(Z'\bar{M}Z)^{-1}Z'\bar{M}Z(Z'\bar{M}Z)^{-1}] \\ &= N\sigma^2(Z'\bar{M}Z)^{-1}. \end{aligned}$$

Remember to remove the N in calculating estimated variance for $\hat{\delta}$. (*Why*?)

Estimation of σ^2:

$$\hat{\sigma}^2 = (y_1 - Z\hat{\delta})'(y_1 - Z\hat{\delta})/N.$$

Note that $Z = [Y_2 X_1]$ appears in the expressions for $\hat{\sigma}^2$, not $[\hat{Y}_2 X_1]$.

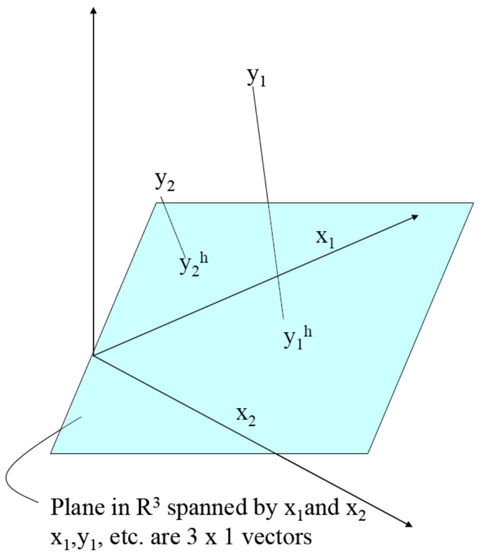
If you regress y_1 on $\hat{Y}_2$ and X_1, you will get the right coefficients but the wrong standard errors.

GEOMETRY OF 2SLS:

Take:
$N = 3$ (observations)
$K = 2$ (exogenous variables),
$K_1 = 1$ (included exogenous variables) and
$G_1 = 2$ (included endogenous variables - one is normalized).

How many parameters?

2SLS



y_1
y_2
x_1
y_2^h
y_1^h
x_2
Plane in R^3 spanned by x_1 and x_2
x_1, y_1, etc. are 3 x 1 vectors

$\hat{Y}_2$ is in the plane spanned by X_1 and X_2. y_1 is projected to the plane spanned by $\hat{Y}_2$ and X_1.

Note that X_1 and X_2 <u>and</u> X_1 and $\hat{Y}_2$ span the same plane. (*Why*?)

Model is just identified (projection of both stages is to the same plane).

What happens if the model is <u>overidentified</u>? (For example, $K_1 = 0$, that is, no included regressors).

What if <u>underidentified</u>? (For example, $K_2 = 2$, that is, no excluded regressors).