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Econ 620 – Spring 2008
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Solution Key for PS #4

1. Suppose that the regression model is;

$$\begin{aligned} y &= X\beta + \varepsilon \\ E(\varepsilon) &= 0, E(\varepsilon\varepsilon') = \sigma^2\Omega \end{aligned}$$

Assume that Ω is known.

(a) the covariance matrix of the OLS estimator

$$\begin{aligned} \widehat{\beta}_{OLS} &= (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'\varepsilon \\ E(\widehat{\beta}_{OLS}) &= \beta \\ Var(\widehat{\beta}_{OLS}) &= E[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}] \\ &= (X'X)^{-1}X'E(\varepsilon\varepsilon')X(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1} \end{aligned}$$

the covariance matrix of the GLS estimator

$$\begin{aligned} \widehat{\beta}_{GLS} &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon \\ E(\widehat{\beta}_{GLS}) &= \beta \\ Var(\widehat{\beta}_{GLS}) &= E[(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon\varepsilon'\Omega^{-1}X(X'\Omega^{-1}X)^{-1}] \\ &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}E(\varepsilon\varepsilon')\Omega^{-1}X(X'\Omega^{-1}X)^{-1} \\ &= \sigma^2(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1} \\ &= \sigma^2(X'\Omega^{-1}X)^{-1} \end{aligned}$$

(b) the covariance matrix of the OLS residual vector

$$\begin{aligned} e &= y - X\widehat{\beta}_{OLS} = (I - X(X'X)^{-1}X')y = My = M\varepsilon \\ E(e) &= ME(\varepsilon) = 0 \end{aligned}$$

Therefore,

$$
\begin{aligned}
Var(e) &= E(ee') = E(M\varepsilon\varepsilon' M) = \sigma^2 M\Omega M \\
&= \sigma^2(I - X(X'X)^{-1}X')\Omega(I - X(X'X)^{-1}X') \\
&= \sigma^2(\Omega - \Omega X(X'X)^{-1}X' - X(X'X)^{-1}X'\Omega \\
&\quad + X(X'X)^{-1}X'\Omega X(X'X)^{-1}X')
\end{aligned}
$$

(c) the covariance matrix of the GLS residual vector

$$
\begin{aligned}
\widetilde{e} &= y - X\widehat{\beta}_{GLS} = (I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})y \\
&= (I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\varepsilon \\
E(\widetilde{e}) &= (I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})E(\varepsilon) = 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Var(\widetilde{e}) &= E(\widetilde{e}\widetilde{e}') \\
&= E[(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\varepsilon\varepsilon'(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})'] \\
&= (I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})E(\varepsilon\varepsilon')(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})' \\
&= \sigma^2(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\Omega(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})' \\
&= \sigma^2[\Omega - X(X'\Omega^{-1}X)^{-1}X']
\end{aligned}
$$

(d) the covariance matrix of the OLS and the GLS residual vectors

$$
\begin{aligned}
e &= (I - X(X'X)^{-1}X')\varepsilon \\
\widetilde{e} &= (I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\varepsilon
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Cov(e, \widetilde{e}) &= E[e\widetilde{e}'] \\
&= E[(I - X(X'X)^{-1}X')\varepsilon\varepsilon'(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})'] \\
&= (I - X(X'X)^{-1}X')E(\varepsilon\varepsilon')(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})' \\
&= \sigma^2(I - X(X'X)^{-1}X')\Omega(I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1})' \\
&= \sigma^2[\Omega - X(X'X)^{-1}X'\Omega]
\end{aligned}
$$

2. (a) $X_t = \rho X_{t-1} + \varepsilon_t$ where $|\rho| < 1$ and $\varepsilon_t \sim i.i.d.(o, \sigma_\varepsilon^2)$
Use Lag Operator, L.

$$
\begin{aligned}
X_t &= \rho L X_t + \varepsilon_t \\
(1 - \rho L)X_t &= \varepsilon_t \\
X_t &= \frac{\varepsilon_t}{1 - \rho L}
\end{aligned}
$$

Note that $|\rho| < 1$. Then we have;

$$X_t = \varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-1} + \cdots$$

Hence,

$$\begin{aligned}
E(X_t) &= 0, \\
Var(X_t) &= E[(\varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-2} + \cdots)^2] \\
&\quad (\varepsilon_t \text{ and } \varepsilon_{t-k} \text{ are uncorrelated, so cross} \\
&\quad \text{product terms are all zero}) \\
&= E(\varepsilon_t^2) + \rho^2 E(\varepsilon_{t-1}^2) + \rho^4 E(\varepsilon_{t-2}^2) + \cdots \\
&= \sigma_\varepsilon^2 \left(\frac{1}{1-\rho^2}\right)
\end{aligned}$$

$$\begin{aligned}
Cov(X_t, X_{t-k}) &= E[(\varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-2} + \cdots) \\
&\quad (\varepsilon_{t-k} + \rho\varepsilon_{t-k-1} + \rho^2\varepsilon_{t-k-2} + \cdots)] \\
&= E(\rho^k\varepsilon_{t-k}^2) + E(\rho^{k+1}\varepsilon_{t-k-1}^2) + E(\rho^{k+2}\varepsilon_{t-k-2}^2) + \cdots \\
&= \rho^k E(\varepsilon_{t-k}^2) + \rho^{k+1} E(\varepsilon_{t-k-1}^2) + \rho^{k+2} E(\varepsilon_{t-k-2}^2) + \cdots \\
&= \sigma_\varepsilon^2 \left(\frac{\rho^k}{1-\rho}\right)
\end{aligned}$$

Therefore, Autocorrelation function $\theta(k)$ is as follows.

$$\begin{aligned}
\theta(k) &= \frac{Cov(X_t, X_{t-k})}{\sqrt{Var(X_t)}\sqrt{Var(X_{t-k})}} \\
&= \frac{Cov(X_t, X_{t-k})}{Var(X_t)} \\
(\text{since } Var(X_t) &= Var(X_{t-k}) \text{ by stationarity}) \\
&= \frac{\sigma_\varepsilon^2 \left(\frac{\rho^k}{1-\rho}\right)}{\sigma_\varepsilon^2 \left(\frac{1}{1-\rho^2}\right)} = \rho^k
\end{aligned}$$

Therefore, this is gives a geometrically declining autocorreltion function and a partial autocorrelation function with zeros for k>1.

(b) $Y_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$ where $\varepsilon_t \sim i.i.d.(o, \sigma_\varepsilon^2)$

$$
\begin{aligned}
E(Y_t) &= 0, \\
Var(Y_t) &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})^2] \\
&\quad (\varepsilon_t \text{ and } \varepsilon_{t-k} \text{ are uncorrelated, so cross} \\
&\quad \text{product terms are all zero}) \\
&= E(\varepsilon_t^2) + {\theta_1}^2 E(\varepsilon_{t-1}^2) + {\theta_2}^2 E(\varepsilon_{t-2}^2) \\
&= \sigma_\varepsilon^2 \left(1 + {\theta_1}^2 + {\theta_2}^2\right) \\
&\Rightarrow \theta(0) = 1 (\text{Autocorrelation function at lag 0})
\end{aligned}
$$

$$
\begin{aligned}
Cov(Y_t, Y_{t-1}) &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})(\varepsilon_{t-1} + \theta_1\varepsilon_{t-2} + \theta_2\varepsilon_{t-3})] \\
&= E(\theta_1\varepsilon_{t-1}^2) + E(\theta_2\theta_1\varepsilon_{t-2}) \\
&= \sigma_\varepsilon^2(\theta_1 + \theta_1\theta_2) \\
\theta(1) &= \frac{Cov(Y_t, Y_{t-1})}{Var(Y_t)} = \frac{\sigma_\varepsilon^2(\theta_1 + \theta_1\theta_2)}{\sigma_\varepsilon^2\,(1 + {\theta_1}^2 + {\theta_2}^2)} \\
&= \frac{\theta_1 + \theta_1\theta_2}{1 + {\theta_1}^2 + {\theta_2}^2}
\end{aligned}
$$

$$
\begin{aligned}
Cov(Y_t, Y_{t-2}) &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})(\varepsilon_{t-2} + \theta_1\varepsilon_{t-3} + \theta_2\varepsilon_{t-4})] \\
&= E(\theta_2\varepsilon_{t-2}^2) \\
&= \sigma_\varepsilon^2\theta_2 \\
\theta(2) &= \frac{Cov(Y_t, Y_{t-2})}{Var(Y_t)} = \frac{\sigma_\varepsilon^2\theta_2}{\sigma_\varepsilon^2\,(1 + {\theta_1}^2 + {\theta_2}^2)} \\
&= \frac{\theta_2}{1 + {\theta_1}^2 + {\theta_2}^2}
\end{aligned}
$$

In addition, using the above method, we can easily know that $\theta(k) = 0$ when k>2.

Hence,

$$
\begin{aligned}
\theta(0) &= 1 \\
\theta(1) &= \frac{\theta_1 + \theta_1\theta_2}{1 + {\theta_1}^2 + {\theta_2}^2} \\
\theta(2) &= \frac{\theta_2}{1 + {\theta_1}^2 + {\theta_2}^2} \\
\theta(k) &= 0, \text{ when k>2}
\end{aligned}
$$

The autocorrelation fuction is zero for k>2 and the pac function dclines geometrically in absolute value(Possibly, it may change signs in terms of real value).

3. The following model is specified:

$$\begin{aligned} y_1 &= \gamma_1 y_2 + \beta_{11} x_1 + \varepsilon_1 \\ y_2 &= \gamma_2 y_1 + \beta_{22} x_2 + \beta_{32} x_3 + \varepsilon_2 \end{aligned}$$

All variables are in measured in deviations from their means. The sample of 25 observations produces the following matrix of sum of squares and cross products:

	y_1	y_2	x_1	x_2	x_3
y_1	20	6	4	3	5
y_2	6	10	3	6	7
x_1	4	3	5	2	3
x_2	3	6	2	10	8
x_3	5	7	3	8	15

We write the model as:

$$\begin{aligned} y_1 &= \gamma_1 y_2 + \beta_{11} x_1 + \epsilon_1 = Z_1 \delta_1 + \epsilon_1 \\ y_2 &= \gamma_2 y_1 + \beta_{22} x_2 + \beta_{32} x_3 + \epsilon_2 = Z_2 \delta_2 + \epsilon_2 \end{aligned}$$

The relevant submatrices are;

$$X'X = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix} \qquad X'y_1 = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \qquad X'y_2 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$$

$$X'Z_1 = \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix} \qquad X'Z_2 = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix}$$

$$Z_1'Z_1 = \begin{pmatrix} 10 & 3 \\ 3 & 5 \end{pmatrix} \qquad Z_2'Z_2 = \begin{pmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix} \qquad Z_1'Z_2 = \begin{pmatrix} 6 & 6 & 7 \\ 4 & 2 & 3 \end{pmatrix}$$

$$Z_1'y_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \qquad Z_1'y_2 = \begin{pmatrix} 10 \\ 3 \end{pmatrix} \qquad Z_2'y_1 = \begin{pmatrix} 20 \\ 3 \\ 5 \end{pmatrix} \qquad Z_2'y_2 = \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix}$$

$$y_1'y_1 = 20 \qquad y_2'y_2 = 10 \qquad y_1'y_2 = 6$$

(a) Estimate the two equations by OLS.

For equation 1,

$$\begin{aligned} \widehat{\delta}_1^{OLS} &= \begin{pmatrix} \widehat{\gamma}_1 \\ \widehat{\beta}_{11} \end{pmatrix} = (Z_1'Z_1)^{-1} Z_1'y_1 \\ &= \begin{pmatrix} 10 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0.122 & -0.0732 \\ -0.0732 & 0.2439 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0.439 \\ 0.5366 \end{pmatrix} \end{aligned}$$

For equation 2,

$$\begin{aligned}
\widehat{\delta}_2^{OLS} &= \begin{pmatrix} \widehat{\gamma}_2 \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{32} \end{pmatrix} = (Z_2'Z_2)^{-1}Z_2'y_2 \\
&= \begin{pmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.0546 & -0.0032 & -0.0165 \\ -0.0032 & 0.1746 & -0.0921 \\ -0.0165 & -0.0921 & 0.1213 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.1930 \\ 0.3841 \\ 0.1975 \end{pmatrix}
\end{aligned}$$

(b) Estimate the parameters of the two equations by 2SLS.

For equation 1,

$$\begin{aligned}
\widehat{\delta}_1^{2SLS} &= \begin{pmatrix} \widehat{\gamma}_1 \\ \widehat{\beta}_{11} \end{pmatrix} = (Z_1'P_xZ_1)^{-1}Z_1'P_xy_1 \\
&= \left[\begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix}' \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix} \right]^{-1} \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix}' \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 0.3688 \\ 0.5787 \end{pmatrix}
\end{aligned}$$

For equation 2,

$$\begin{aligned}
\widehat{\delta}_2^{2SLS} &= \begin{pmatrix} \widehat{\gamma}_2 \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{32} \end{pmatrix} = (Z_2'P_xZ_2)^{-1}Z_2'P_xy_2 \\
&= \begin{pmatrix} 0.484375 \\ 0.367188 \\ 0.109375 \end{pmatrix}
\end{aligned}$$

4. The following model is specified:

$$\begin{aligned}
y_1 &= \gamma_1 y_2 + \beta_{11}x_1 + \beta_{12}x_2 + \beta_{13}x_3 + \varepsilon_1 \\
y_2 &= \gamma_2 y_1 + \beta_{21}x_1 + \beta_{22}x_2 + \beta_{23}x_3 + \varepsilon_2
\end{aligned}$$

The error terms have both expectation zero. We consider only exclusion restrictions. Using the order and rank conditions, verify whether the model is identified under the following restrictions:

In the lecture note 15, we have the following conditions:

Order condition: There is at least one blank space in the row of the identified eqution. This is a necessary condition.

Rank condition: The variable left out of the equation considered must appear in the other. This is a necessary condition.

The followings are all about checking order condition.

(a) $\beta_{12} = \beta_{13} = \beta_{21} = 0$

	y1	y2	x1	x2	x3
eq.1(y1)	$\curlyvee$	$\curlyvee$	$\curlyvee$		
eq.2(y2)	$\curlyvee$	$\curlyvee$		$\curlyvee$	$\curlyvee$

The order condition is satisfied. Eq.1 is over-identified, and Eq. 2 is just-identified.

(b) $\beta_{11} = \beta_{12} = \beta_{13} = 0$

	y1	y2	x1	x2	x3
eq.1(y1)	$\curlyvee$	$\curlyvee$			
eq.2(y2)	$\curlyvee$	$\curlyvee$	$\curlyvee$	$\curlyvee$	$\curlyvee$

The ordr condition is not satisfied. Eq.1 is over-identified. But Eq. 2 is under-identified.

(c) $\beta_{13} = \beta_{22} = 0$

	y1	y2	x1	x2	x3
eq.1(y1)	$\curlyvee$	$\curlyvee$	$\curlyvee$	$\curlyvee$	
eq.2(y2)	$\curlyvee$	$\curlyvee$	$\curlyvee$		$\curlyvee$

The order condition is satisfied. Both equations are just-identified.

(d) $\beta_{11} = \beta_{12} = 0$

	y1	y2	x1	x2	x3
eq.1(y1)	$\curlyvee$	$\curlyvee$			$\curlyvee$
eq.2(y2)	$\curlyvee$	$\curlyvee$	$\curlyvee$	$\curlyvee$	$\curlyvee$

The order condition is not satisfied. Eq.1 is over-identified. But Eq. 2 is under-identified.

5. (2005 Final) You have a regression model $y_i = \alpha + \beta x_i + \varepsilon_i$ where x is either 0 or 1. In an attempt to simplify your estimation problem, you calculate $\overline{y}_0$ and $\overline{y}_1$, wher these are the sample means corresponding to observations with x=o and x=1 respectively. Then you calculate α^* and β^* by $\overline{y}_0$ and $\overline{y}_1 - \overline{y}_0$. Are your estimators unbiased? Consistent? Efficient(minimum variance unbiased)?

   We can rewrite the model as $y_i = \alpha(1 - x_i) + (\beta + \alpha)x_i + \varepsilon_i$. Then OLS estimators $\widehat{\alpha} = \sum y_i 1[x_i = 0]/\sum 1[x_i = 0] = y_0$ and $(\widehat{\beta} + \widehat{\alpha}) = \overline{y}_1$. Therefore, α^* and β^* are equal to the OLS estimators. They are unbiased, consistent and efficient.