Cornell University
Department of Economics

Econ 620 – Spring 2008
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Selective Solutions for PS # 3

1. (Refer to the solution for (2) from 2006 midterm)
2. (Refer to the solution for (1) from 2002 midterm)
3.

$$\begin{aligned} \text{Model I} &: \quad y = X\beta + \varepsilon = X_1\beta_1 + X_2\beta_2 + \varepsilon \\ \text{Model II} &: \quad M_2 y = M_2 X_1 \beta_1 + \varepsilon \end{aligned}$$

   Model I: $\widehat{\beta}_1^I = (X_1' M_2 X_1)^{-1} X_1' M_2 y$

   Model II: $\widehat{\beta}_2^{II} = [(M_2X_1)'(M_2X_1)]^{-1}(M_2X_1)'y$

   Recall that M_2 is idempotent and symmetric, then we have

   $\widehat{\beta}_2^{II} = (X_1{}' M_2' M_2 X_1)^{-1} X_1{}' M_2' y = (X_1{}' M_2 X_1)^{-1} X_1{}' M_2 y$

   Therefore, we have $\widehat{\beta}_1^I = \widehat{\beta}_2^{II}$

4. For the standard normal regression model;

$$y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I)$$

   (a) Write down the log-likelihood function. And find MLE for β and σ^2.
   Log-likelihood function:

$$l(\beta, \sigma^2) = -\frac{N}{2}\log 2\pi - \frac{N}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)$$

   For MLE, find the First Order Condition(Score function)

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= -\frac{1}{2\sigma^2}(-2X'y + 2X'X\beta) = 0 \\ \frac{\partial l}{\partial \sigma^2} &= -\frac{N}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}(y - X\beta)'(y - X\beta) = 0 \end{aligned}$$

   From the above, we can obtain MLE.

$$\begin{aligned} \widehat{\beta}_{ML} &= (X'X)^{-1}X'y \\ \widehat{\sigma^2}_{ML} &= \frac{e'e}{N},\ \text{where } e = y - X\widehat{\beta}_{ML} \end{aligned}$$

(b) Find the asymptotic distribution of MLE.

Recall that $\sqrt{N}(\widehat{\theta} - \theta_0) \xrightarrow{d} N(0, i(\theta_0)^{-1})$ and that $i(\theta_0) = -h(\theta_0)$ where $h(\theta_0)$ is expected hessian.

For elements of hessian, find second derivatives of log likelihood function

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta\partial\beta'} &= -\frac{X'X}{\sigma^2} \\
\frac{\partial^2 l}{\partial(\sigma^2)^2} &= \frac{N}{2}\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}(y - X\beta)'(y - X\beta) \\
\frac{\partial^2 l}{\partial\beta\partial\sigma^2} &= \frac{1}{2(\sigma^2)^2}(-2X'y + 2X'X\beta)
\end{aligned}$$

From the above, the expected hessian is

$$\begin{aligned}
h(\theta_0) &= \begin{bmatrix} -E(\frac{X'X}{N\sigma^2}) & E[\frac{1}{N2(\sigma^2)^2}(-2X'y + 2X'X\beta)] \\ E[\frac{1}{N2(\sigma^2)^2}(-2X'y + 2X'X\beta)]' & E[\frac{1}{N}(\frac{N}{2}\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}(y - X\beta)'(y - X\beta))] \end{bmatrix} \\
&= \begin{bmatrix} -E(\frac{X'X}{N\sigma^2}) & 0 \\ 0 & -\frac{1}{2(\sigma^2)^2} \end{bmatrix}
\end{aligned}$$

Therefore, we have

$$i(\theta_0) = \begin{bmatrix} E(\frac{X'X}{N\sigma^2}) & 0 \\ 0 & \frac{1}{2(\sigma^2)^2} \end{bmatrix}$$

$$i(\theta_0)^{-1} = \begin{bmatrix} \sigma^2[E(\frac{X'X}{N})]^{-1} & 0 \\ 0 & 2(\sigma^2)^2 \end{bmatrix}$$

Finally, the asymptotic distribution will be as follows.

$$\sqrt{N}\begin{bmatrix} \widehat{\beta} - \beta \\ \widehat{\sigma^2} - \sigma^2 \end{bmatrix} \xrightarrow{d} N(0, \begin{bmatrix} \sigma^2[E(\frac{X'X}{N})]^{-1} & 0 \\ 0 & 2(\sigma^2)^2 \end{bmatrix}$$

(c) Show that

$$E(-\frac{\partial^2 LogL}{\partial\beta\partial\beta'}) = E([\frac{\partial LogL}{\partial\beta}][\frac{\partial LogL}{\partial\beta'}]')$$

LHS: $E(\frac{X'X}{\sigma^2})$

RHS:

$$
\begin{aligned}
& E[\{\frac{1}{2(\sigma^2)}(-2X'y+2X'X\beta)\}\{\frac{1}{2(\sigma^2)}(-2X'y+2X'X\beta)\}'] \\
= \; & E[\frac{1}{4(\sigma^2)^2}(-2X'y+2X'X\beta)(-2X'y+2X'X\beta)'] \\
= \; & E[\frac{1}{4(\sigma^2)^2}(4X'(y-X\beta)(y-X\beta)'X)] \\
= \; & E[\frac{1}{4(\sigma^2)^2}4X'E\{(y-X\beta)(y-X\beta)' \mid X\}X] \text{ by law of iteative expectation} \\
= \; & E[\frac{1}{4(\sigma^2)^2}4X'E\{\varepsilon\varepsilon' \mid X\}X] \\
(E\{\varepsilon\varepsilon' \mid \; & X\} = \sigma^2 I) \\
= \; & E[\frac{1}{\sigma^2}X'X]
\end{aligned}
$$

Hence, we have RHS=LHS.

5. Consider the following regression model;

$$y = X\beta + \varepsilon, \; \varepsilon \sim N(0, \sigma^2 I)$$

with $E(\varepsilon) = 0, E(\varepsilon\varepsilon') = \sigma^2 I$. Three potential linear estimatiors for β are

$$
\begin{aligned}
\widehat{\beta} &= (X'X)^{-1}X'y \\
\widetilde{\beta} &= \widehat{\beta} + N^{-1}1 \\
\overline{\beta} &= \widehat{\beta} + N^{-\frac{1}{2}}1
\end{aligned}
$$

where 1 is a $k \times 1$ vector of ones.

(a) Which of these are unbiased?
=> $\widehat{\beta}$ (done in the section)

(b) Which are consistent?
=> They are all consistent (done in the section)

(c) What are the asymptotic distributions of $\sqrt{N}(\widehat{\beta} - \beta), \sqrt{N}(\widetilde{\beta} - \beta)$, and $\sqrt{N}(\overline{\beta} - \beta)$?
i) $\sqrt{N}(\widehat{\beta} - \beta) \sim N(0, \sigma^2 Q^{-1})$, where $Q = p\lim(\frac{X'X}{N})$ (done in the section)

ii) $\sqrt{N}(\widetilde{\beta}-\beta)$:

$$\begin{aligned}
\widetilde{\beta} &= (X'X)^{-1}X'(X\beta+\varepsilon)+\frac{1}{N} \\
&= \beta+(X'X)^{-1}X'\varepsilon+\frac{1}{N} \\
&\quad \text{Therefore,} \\
\sqrt{N}(\widetilde{\beta}-\beta) &= (\frac{X'X}{N})^{-1}(\frac{X'\varepsilon}{\sqrt{N}})+\frac{1}{\sqrt{N}} \\
\text{Recall that } \frac{1}{\sqrt{N}} &= o_p(1), p\lim(\frac{X'X}{N})^{-1}=Q^{-1}, (\frac{X'\varepsilon}{\sqrt{N}}) \xrightarrow{d} N(0,\sigma^2 Q) \\
&\quad \text{Hence,} \\
\sqrt{N}(\widetilde{\beta}-\beta) &\sim N(0,\sigma^2 Q^{-1})
\end{aligned}$$

iii) $\sqrt{N}(\overline{\beta}-\beta)\sim N(1,\sigma^2Q^{-1})$ (done in the section)

6. Suppose $x_i, i=1,2,\cdots$ is a sequence of independent random variables where each x_i is uniformly distributed with density

$$f(x_i)=1_{1[0\leq x_i<1]} \text{ for all } i$$

(a) Find $p\lim \frac{1}{n}\sum_{i=1}^n x_i, p\lim \frac{1}{n}\sum_{i=1}^n x_i^2$ and $p\lim \frac{1}{n}\sum_{i=1}^n x_i^3$
Let's apply the law of large numbers.
$p\lim \frac{1}{n}\sum_{i=1}^n x_i = E(x_i)=\int_0^1 x\cdot 1dx=\frac{1}{2}$
$p\lim \frac{1}{n}\sum_{i=1}^n x_i^2 = E(x_i^2)=\int_0^1 x^2\cdot 1dx=\frac{1}{3}$
$p\lim \frac{1}{n}\sum_{i=1}^n x_i^3 = E(x_i^3)=\int_0^1 x^3\cdot 1dx=\frac{1}{4}$

(b) Suppose $x_i's$ are as above and $y_i=x_i^2+\varepsilon_i$ with ε_i independent of x_i and $E(\varepsilon_i)=0, Var(\varepsilon_i)=\sigma^2$. You run the regression $Ey_i=\alpha+\beta x_i$. Find $p\lim\widehat{\alpha}$ and $p\lim\widehat{\beta}$ where $\widehat{\alpha}$ and $\widehat{\beta}$ are the least squares estimators.

$$
\begin{aligned}
\widehat{\beta} &= \frac{\sum(x_i-\overline{x})y_i}{\sum(x_i-\overline{x})^2} = \frac{\sum(x_i-\overline{x})(x_i^2+\varepsilon_i)}{\sum x_i^2 - n(\overline{x})^2} \\
&= \frac{\sum x_i^3 - \overline{x}\sum x_i^2 + \sum x_i\varepsilon_i - \overline{x}\sum\varepsilon_i}{\sum x_i^2 - N(\overline{x})^2} \\
&= \frac{\frac{\sum x_i^3}{N} - \overline{x}\frac{\sum x_i^2}{N} + \frac{\sum x_i\varepsilon_i}{N} - \overline{x}\frac{\sum\varepsilon_i}{N}}{\frac{\sum x_i^2}{N} - (\overline{x})^2} \\
&\text{Therefore,} \\
p\lim\widehat{\beta} &= \frac{p\lim\frac{\sum x_i^3}{N} - p\lim\overline{x}\cdot p\lim\frac{\sum x_i^2}{N} + p\lim\frac{\sum x_i\varepsilon_i}{N} - p\lim\overline{x}\cdot p\lim\frac{\sum\varepsilon_i}{N}}{p\lim\frac{\sum x_i^2}{N} - p\lim(\overline{x})^2} \\
&= \frac{E(x_i^3) - E(x_i)\cdot E(x_i^2) + E(x_i\varepsilon_i) - E(x_i)\cdot E(\varepsilon_i)}{E(x_i^2) - (E(x_i))^2} \\
&= \frac{\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3} + 0 - \frac{1}{2}\cdot 0}{\frac{1}{3} - (\frac{1}{2})^2} \\
&= 1
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\widehat{\alpha} &= \overline{y} - \widehat{\beta}\overline{x} = \frac{\sum y_i}{N} - \widehat{\beta}\frac{\sum x_i}{N} = \frac{\sum(x_i^2+\varepsilon_i)}{N} - \widehat{\beta}\frac{\sum x_i}{N} \\
&\text{Therefore,} \\
p\lim\widehat{\alpha} &= p\lim\frac{\sum x_i^2}{N} + p\lim\frac{\sum\varepsilon_i}{N} - p\lim\widehat{\beta}\cdot p\lim\frac{\sum x_i}{N} \\
&= \frac{1}{3} + 0 - 1\cdot\frac{1}{2} = -\frac{1}{6}
\end{aligned}
$$