Cornell University
Department of Economics

Econ 620 – Spring 2008
Instructor: Professor Kiefer
TA: Jae-Ho Yun (jy238@cornell.edu)

Solutions for PS # 2

1. (a) $X^2 \tilde{} \chi^2(1)$
   (b) $N(0,1)$
   (c) $\chi^2(1)$ (by the continuous mapping theorem)

2.
$$Z'Z = \begin{pmatrix} 760 & 30 & 1300 \\ 30 & 31 & 0 \\ 1300 & 0 & 2480 \end{pmatrix}$$

From the above matrix, we can know that
$n = 31, \sum x_i = 0, \sum y_i = 30, \sum x_i^2 = 2480, \sum y_i^2 = 760, \sum x_i y_i = 1300$
Therefore, we have sample means for x and y as follows.
$\overline{x} = 0, \overline{y} = 30/31$

i) sample size: 31

ii) the mean of X: $0 \div 31 = 0$

iii) We know that in the two variable case R^2 is the square of the correlation between X and Y. Therefore, we're going to use the following formula

$$\begin{aligned} r &= \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum(x_i - \overline{x})^2}\sqrt{\sum(y_i - \overline{y})^2}} \\ &= \frac{\sum x_i y_i - n\overline{x}\overline{y}}{\sqrt{\sum x_i^2 - n\overline{x}^2}\sqrt{\sum y_i^2 - n\overline{y}^2}} \\ &= \frac{1300 - 31 \times 0 \times 30/31}{\sqrt{2480 - 31 \times 0^2}\sqrt{760 - 31 \times (30/31)^2}} \\ &= 0.9655 \\ R^2 &= r^2 = 0.9323 \end{aligned}$$

iv) OLS estimator

$$\begin{aligned}\widehat{\beta} &= \frac{\sum(x_i-\overline{x})(y_i-\overline{y})}{\sum(x_i-\overline{x})^2} = \frac{\sum x_i y_i - n\overline{x}\overline{y}}{\sum x_i^2 - n\overline{x}^2} \\ &= \frac{1300 - 31\times 0\times 30/31}{2480 - 31\times 0^2} \\ &= 0.5242\end{aligned}$$

v) F-statistics for $\alpha = 0, \beta = 0$

For this, we have to calculate $S^2 (= \frac{e'e}{n-k} = \frac{e'e}{29})$

$$\begin{aligned}e'e &= \sum(y_i-\overline{y})^2 - \widehat{\beta}^2\sum(x_i-\overline{x})^2 = \sum y_i^2 - n\overline{y}^2 - 0.5242^2(\sum x_i^2 - n\overline{x}^2) \\ &= 760 - 31\times(30/31)^2 - 0.5242^2(2480 - 31\times 0^2) \\ &= 49.50 \\ S^2 &= 49.50/29 = 1.71\end{aligned}$$

For F-statistics, we want to use the following formula(Refer to the lecture note 5. However, $*$ implies restricted model here). Note that $e^{*\prime}e^* = \sum y_i^2$.(Since there is no explanatory variable left under the null hypothesis)

$$\begin{aligned}F &= \frac{(e^{*\prime}e^* - e'e)/(trM^* - trM)}{e'e/trM} \\ &= \frac{(760-49.50)/2}{1.71} \\ &= 207.7485\end{aligned}$$

vi) $Ex = \alpha^* + \beta^* y$

R^2 is same as before.

$$\begin{aligned}\widehat{\beta}^* &= \frac{\sum(x_i-\overline{x})(y_i-\overline{y})}{\sum(y_i-\overline{y})^2} = \frac{\sum x_i y_i - n\overline{x}\overline{y}}{\sum y_i^2 - n\overline{y}^2} \\ &= \frac{1300 - 31\times 0\times 30/31}{760 - 31\times(30/31)^2} \\ &= 1.78\end{aligned}$$

$$
\begin{aligned}
\widetilde{e}'\widetilde{e} &= \sum(x_i-\overline{x})^2-\widehat{\beta}^{*2}\sum(y_i-\overline{y})^2=\sum x_i^2-n\overline{x}^2-1.78^2(\sum y_i^2-n\overline{y}^2)\\
&= 2480-31\times 0^2-1.78^2(760-31\times(30/31)^2)\\
&= 164.00\\
\widetilde{S}^2 &= 164.00/29=5.66\\
s.e.(\widehat{\beta}^*) &= \sqrt{5.66}*\left(\sqrt{\sum y_i^2-n\overline{y}^2}\right)^{-1}=\sqrt{5.66}*\left(\sqrt{760-31\times(30/31)^2}\right)^{-1}\\
&= 0.0880\\
t &= \widehat{\beta}^*/s.e.(\widehat{\beta}^*)=1.78/0.0880=20.23
\end{aligned}
$$

(a) $y=X_1\beta_1+X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'M_1X_2)^{-1}X_2'M_1y$

(b) $P_1y=X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'X_2)^{-1}X_2'P_1y$

(c) $P_1y=P_1X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'P_1X_2)^{-1}X_2'P_1y$

(d) $M_1y=X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'X_2)^{-1}X_2'M_1y$

(e) $y=M_1X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'M_1X_2)^{-1}X_2'M_1y$

(f) $M_1y=M_1X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'M_1X_2)^{-1}X_2'M_1y$

(g) $M_1y=X_1\beta_1+M_1X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'M_1X_2)^{-1}X_2'M_1y$

(h) $M_1y=M_1X_1\beta_1+M_1X_2\beta_2+\varepsilon \Rightarrow \widehat{\beta_2}=(X_2'M_1X_2)^{-1}X_2'M_1y$

Therefore, e,f,g,h give the same results.

3. Unbiasedness implies that $c_1+c_2=1$, since $E(b)=(c_1+c_2)\beta$.

   Therefore, the problem consists of minimizing $Var(b)=c_1^2v_1+c_2^2v_2$ subject to $c_1+c_2=1$.

   Second order conditions will be met, since the objective function is quadratic and the restriction is linear and from the first order conditions we can conclude that $c_1=\frac{v_2}{v_1+v_2}$ and $c_2=\frac{v_1}{v_1+v_2}$.

   (a) $E[b\mid X]=\beta+(X'X)^{-1}X'E[\varepsilon\mid X]=\beta$ since $E[\varepsilon\mid X]=0$.
   Therefore, $E[b]=E(E[b\mid X]\mid X)=E(\beta\mid X)=\beta$.
   Hence, if the regressors are stochastic, as long as they are uncorrelated with the error term (and the all other assumptions hold), the OLS estimator of β is still unbiased.

   (b) $Var(b)=E_X[Var(b\mid X)]+Var_XE[b\mid X]$
   Note that $Var(b\mid X)=\sigma^2(X'X)^{-1}$ (random varable, since X is random)
   and $E[b\mid X]=\beta$ (constant)
   Therefore,
   $Var(b)=E_X[\sigma^2(X'X)^{-1}]+Var_X[\beta]=\sigma^2E[(X'X)^{-1}]+0=\sigma^2E[(X'X)^{-1}]$

$$
\begin{aligned}
X_n &= 3 - \frac{1}{n^2} \\
Y_n &= \sqrt{n}\frac{\overline{Z_n}}{\sigma}
\end{aligned}
$$

where $\overline{Z_n} = \frac{1}{n}\sum_{i=1}^{n} Z_i$ and Z_i's are i.i.d. with mean zero and variance σ^2. Note that $X_n - 3 = o_p(1)$ ($p\lim X_n = 3$) and $Y_n \xrightarrow{d} N(0,1)$.

(a) $X_n + Y_n \xrightarrow{d} N(3,1)$

(b) $X_n Y_n \xrightarrow{d} N(0, 3^2)$

(c) $Y_n^2 \xrightarrow{d} \chi^2(1)$

(a) Prove that $\widehat{\beta} \xrightarrow{p} \beta$.(Using Law of Large Numbers)

$$
\begin{aligned}
\widehat{\beta} &= \beta + (\frac{X'X}{N})^{-1}(\frac{X'\varepsilon}{N}) \\
p\lim \widehat{\beta} &= \beta + p\lim(\frac{X'X}{N})^{-1} p\lim(\frac{X'\varepsilon}{N}) \\
&= \beta + Q^{-1} \cdot 0 \\
&= \beta
\end{aligned}
$$

(b) Find the asymptotic distribution of $\sqrt{N}(\widehat{\beta} - \beta)$.

$$
\begin{aligned}
\widehat{\beta} &= \beta + (\frac{X'X}{N})^{-1}(\frac{X'\varepsilon}{N}) \\
\sqrt{N}(\widehat{\beta} - \beta) &= (\frac{X'X}{N})^{-1}(\frac{X'\varepsilon}{\sqrt{N}})
\end{aligned}
$$

Note that $p\lim(\frac{X'X}{N})^{-1} = Q^{-1}$ by Law of Large Numbers and Continuous Mapping Theorem and that $(\frac{X'\varepsilon}{\sqrt{N}}) \xrightarrow{d} N(0, \sigma^2 Q)$ by Central Limit Theorem.

Then, we have the following limiting distribution.

$$
\begin{aligned}
\sqrt{N}(\widehat{\beta} - \beta) &= (\frac{X'X}{N})^{-1}(\frac{X'\varepsilon}{\sqrt{N}}) \\
&\xrightarrow{d} N(0, \sigma^2 Q^{-1} Q Q^{-1}) \\
&= N(0, \sigma^2 Q^{-1})
\end{aligned}
$$

(c) Prove that $p\lim S^2 = \sigma^2$ where $S^2 = \frac{e'e}{N-k}$

$$
\begin{aligned}
S^2 &= \frac{e'e}{N-K} = \frac{\varepsilon' M \varepsilon}{N-K} \\
&= \frac{\varepsilon'(I - X(X'X)^{-1}X')\varepsilon}{N-K} \\
&= \frac{N}{N-K}\left[\frac{\varepsilon'\varepsilon}{N} - \frac{\varepsilon' X (X'X)^{-1} X'\varepsilon}{N}\right] \\
&= \frac{N}{N-K}\left[\frac{\varepsilon'\varepsilon}{N} - \frac{\varepsilon' X}{N}\left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{N}\right]
\end{aligned}
$$

Now, apply the Law of Large Numbers.

$$
\begin{aligned}
p\lim S^2 &= p\lim \frac{N}{N-K}\left[p\lim \frac{\varepsilon'\varepsilon}{N} - p\lim \frac{\varepsilon' X}{N} \cdot p\lim \left(\frac{X'X}{N}\right)^{-1} \cdot p\lim \frac{X'\varepsilon}{N}\right] \\
&= 1 \cdot \left[\sigma^2 - 0 \cdot Q^{-1} \cdot 0\right] \\
&= \sigma^2
\end{aligned}
$$