Econ 620 Spring 2006
Professor N. Kiefer
TA H. Choi

Suggested Solutions to the midterm exam

1. (30pts) Let A be a projection operator on to the orthogonal space to the space spanned by a column of ones.

(a) $R^2 = R_1^2 = R_2^2 = (x'Ay)^2/\{(x'Ax)(y'Ay)\}$

(**Geometry**: It's from $R^2 = \cos^2\theta_0$, where θ_0 is the angle between Ay and Ax.)

(b) $\hat{\beta}_1 = (x'Ay)/(x'Ax)$, $\hat{\beta}_2 = (x'Ay)/(y'Ay)$, and we have $\hat{\beta}_1\hat{\beta}_2 = R^2$.

(c) From $\hat{\beta}_1 = (x'Ay)/(x'Ax)$ and $R^2 = 1 - e'e/(y'Ay)$, we have

$$t_1 = \hat{\beta}_1/\sqrt{(x'Ax)^{-1}(e_1'e_1)/(n-2)} = (x'Ay)/\sqrt{(x'Ax)(y'Ay)(1-R^2)/(n-2)}.$$

This proves $t_1 = t_2$. (**Geometry**: It's from (i) $\hat{\beta}_1$ and $\hat{\beta}_2$ have a same sign as shown in (b), and (ii) $t_1^2 = t_2^2 = (n-2)\cot^2\theta_1$, where θ_1 is the angle between Ay and Ax.)

2. (50pts) Let $\sum = \sum_{i=1}^n$.

(a) $l(\theta) = \sum d_i \log\theta + \sum(1-d_i)\log(1-\theta)$, $s(\theta) = \sum d_i/\theta - \sum(1-d_i)/(1-\theta)$, $h(\theta) = -\sum d_i/\theta^2 - \sum(1-d_i)/(1-\theta)^2$, $E(h(\theta)) = -n\{1/\theta + 1/(1-\theta)\} = -n\{1/(\theta(1-\theta))\}$, $I(\theta) = n\{1/(\theta(1-\theta))\}$.

(b) $\hat{\theta}_{ML} = \arg\max l(\theta) = \sum d_i/n$, $\sqrt{n}(\hat{\theta}_{ML} - \theta) \xrightarrow{d} N(0, i(\theta)^{-1})$, where the asymptotic variance $i(\theta) = I(\theta)/n = 1/(\theta(1-\theta))$.

(c) Use $LR = 2(l(\hat{\theta}_{ML}) - l(\theta_0))$, $LM = s(\theta_0)I(\theta_0)^{-1}s(\theta_0)$, or $W = (\hat{\theta}_{ML} - \theta_0)I(\hat{\theta}_{ML})(\hat{\theta}_{ML} - \theta_0)$ with approximating distribution $\chi^2(1)$.

(d) Consider the test,

"Choose H_0 if $\hat{\theta}_{ML} = 0$, and H_A if $\hat{\theta}_{ML} \neq 0$" (Possible tests are randomized tests with this test and a trivial test, "Always reject and choose H_A".)

(e) Type I error $P(Reject\ |\theta = 0) = 0$, and type II error $P(Fail\ to\ Reject|\theta = 1/2) = 0.5$. For the randomized test using the test "Choose H_0 if $\hat{\theta}_{ML} = 0$, and H_A if $\hat{\theta}_{ML} \neq 0$" with probability α, or "Always reject and choose H_A" with probability $(1-\alpha)$, we have $P(Reject\ |\theta = 0) = \alpha$, and $P(Fail\ to\ Reject|\theta = 1/2) = 0.5(1-\alpha)$.

3. (20pts) Let $\sum = \sum_{i=1}^n$.

(a) $X^2 \sim \chi^2(1)$. (b) $X_n \xrightarrow{d} N(0,1)$. (c) $Y_n = X_n^2 \xrightarrow{d} \chi^2(1)$ by the continuous mapping theorem.