Econ 620 Spring 2005
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**Suggested Solutions to the midterm exam**

1. We have $Z = XR$, where

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore we have $R\alpha = \beta$ from $Ey = Z\alpha = XR\alpha = X\beta$. This implies $R\hat{a}_{LS} = \hat{\beta}_{LS}$.

2. We have $\hat{\beta}_2$ for a) $\left(X_2'M_1X_2\right)^{-1}X_2'M_1y$, b) $\left(X_2'X_2\right)^{-1}X_2'P_1y$, c) $\left(X_2'P_1X_2\right)^{-1}X_2'P_1y$, d) $\left(X_2'X_2\right)^{-1}X_2'M_1y$, e) $\left(X_2'M_1X_2\right)^{-1}X_2'M_1y$, f) $\left(X_2'M_1X_2\right)^{-1}X_2'M_1y$, g) $\left(X_2'M_1X_2\right)^{-1}X_2'M_1y$, h) $\left(X_2'M_1X_2\right)^{-1}X_2'M_1y$. Hence we have 4 different estimators. (a)-(d) are all different and (e)-(h) are the same as (a).

3. The prediction $y_p$ is $10\times 1$ vector and has the form $y_p = Cy$, where $y = (y_1, ..., y_n)'$. Under LS assumption, $y_p$ is given by the relationship $Cy = X_p\hat{\beta}$, where $X_p$ is known and $\hat{\beta}$ is OLS estimator. We have $C = X_p\left(X'X\right)^{-1}X'$. The variance of the prediction errors is $\mathrm{Var}(C\varepsilon - \varepsilon_p) = C\mathrm{Var}(\varepsilon)\,C' + Var\left(\varepsilon_p\right) = \sigma^2\left(CC' + I\right) = \sigma^2\left(X_p\left(X'X\right)^{-1}X_p + I\right)$, where $I$ is $10\times 10$ identity matrix. For correlated errors assumption, from the unbiasedness we have $CX = X_p$ and the variance of the prediction error is

$$\begin{aligned} Var(C\varepsilon - \varepsilon_p) &= CVar\left(\varepsilon\right)C' + Var\left(\varepsilon_p\right) - C\ Cov\left(\varepsilon, \varepsilon_p\right) - Cov\left(\varepsilon_p, \varepsilon\right)\ C' \\ &= C\Omega C' + V - CW - W'C', \end{aligned} \tag{1}$$

where $\Omega = Var\left(\varepsilon\right),\ V = Var\left(\varepsilon_p\right)$, and $W = Cov\left(\varepsilon, \varepsilon_p\right)$. We minimize tr$\{Var(C\varepsilon - \varepsilon_p)\}$ with respect to $C$ with the restriction $CX = X_p$. The solution $C^*$ gives us the similar result with a single $y_p$ prediction case. We have

$$y_p = X_p\hat{\beta}_{GLS} + W'\Omega^{-1}\left(y - X\hat{\beta}_{GLS}\right).$$

The last term is the conditional mean of $e_p$ given $e$. Therefore

$$C = X_p\left(X'\Omega^{-1}X\right)^{-1}X'\Omega^{-1} + W\Omega^{-1}\left(I - X\left(X'\Omega^{-1}X\right)^{-1}X'\Omega^{-1}\right),$$

use this in the equation (1) to get the prediction error.

4.

$$\begin{aligned} \hat{\beta}_2 &= \left(\sum\left(1/t - \bar{t}\right)^2\right)^{-1}\sum\left(1/t - \bar{t}\right)\left(y_t - \bar{y}\right) = \left(\sum\left(1/t - \bar{t}\right)^2\right)^{-1}\sum\left(1/t - \bar{t}\right)\left(\beta_2\left(1/t - \bar{t}\right) + \varepsilon_t - \bar{\varepsilon}\right) \\ &= \beta_2 + \left(\sum\left(1/t - \bar{t}\right)^2\right)^{-1}\sum\left(1/t - \bar{t}\right)\left(\varepsilon_t - \bar{\varepsilon}\right), \end{aligned}$$

where $\sum = \sum_{t=1}^{T},\ \bar{t} = T^{-1}\sum\left(1/t\right),\ \bar{\varepsilon} = T^{-1}\sum\varepsilon_i$ and $\bar{y} = T^{-1}\sum y_i$. Noting that plim $\bar{t} = 0$, plim $\bar{\varepsilon} = 0$ and $\sum\left(1/t^2\right) = \pi^2/6 < \infty$, we have plim $\hat{\beta}_2 = \beta_2 + \left(\pi^2/6\right)^{-1}\text{plim}\sum\left(1/t\right)\varepsilon_t$. Since $lim_{T\to\infty}$ $\mathrm{Var}(\sum\left(1/t\right)\varepsilon_t) = lim_{T\to\infty}\sum\left(1/t^2\right) = \pi^2/6 \neq 0$, $\hat{\beta}_2$ is not consistent and plim$\sum\left(1/t\right)\varepsilon_t$ does not exist. The LS estimator $\hat{\beta}_2$ is unbiased. So $E\left(\hat{\beta}_2\right) = \beta$ and $\mathrm{Var}\left(\hat{\beta}_2\right) = \sigma^2\left(X_2'M_1X_2\right)^{-1} = \left(\sum\left(1/t - \bar{t}\right)^2\right)^{-1}$, which implies $\hat{\beta}_2 \sim N\left(\beta, \left(\sum\left(1/t - \bar{t}\right)^2\right)^{-1}\right)$. The asymptotic distribution of $\hat{\beta}_2$ is not given by the usual $\sqrt{T}$-asymptotics, but we have $\left(\hat{\beta}_2 - \beta\right) \xrightarrow{d} N(0, \left(6/\pi^2\right))$.