Cornell University
Department of Economics

Econon 620
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Solution to Problem set # 8

1) We write the model as:

$$y_1 = \gamma_1 y_2 + \beta_{11} x_1 + \epsilon_1 = Z_1 \delta_1 + \epsilon_1$$
$$y_2 = \gamma_2 y_1 + \beta_{22} x_2 + \beta_{32} x_3 + \epsilon_2 = Z_2 \delta_2 + \epsilon_2$$

The relevant submatrices are:

$$X'X = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix} \qquad X'y_1 = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \qquad X'y_2 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$$

$$X'Z_1 = \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix} \qquad X'Z_2 = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix}$$

$$Z_1'Z_1 = \begin{pmatrix} 10 & 3 \\ 3 & 5 \end{pmatrix} \qquad Z_2'Z_2 = \begin{pmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix} \qquad Z_1'Z_2 = \begin{pmatrix} 6 & 6 & 7 \\ 4 & 2 & 3 \end{pmatrix}$$

$$Z_1'y_1 = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \qquad Z_1'y_2 = \begin{pmatrix} 10 \\ 3 \end{pmatrix} \qquad Z_2'y_1 = \begin{pmatrix} 20 \\ 3 \\ 5 \end{pmatrix} \qquad Z_2'y_2 = \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix}$$

$$y_1'y_1 = 20 \qquad y_2'y_2 = 10 \qquad y_1'y_2 = 6$$

a) For equation 1,

$$\begin{aligned} \widehat{\delta}_1^{OLS} = \begin{pmatrix} \widehat{\gamma}_1 \\ \widehat{\beta}_{11} \end{pmatrix} &= (Z_1'Z_1)^{-1} Z_1'y_1 \\ &= \begin{pmatrix} 10 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0.122 & -0.0732 \\ -0.0732 & 0.2439 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0.439 \\ 0.5366 \end{pmatrix} \end{aligned}$$

For equation 2,

$$
\begin{aligned}
\widehat{\delta}_2^{OLS} &= \begin{pmatrix} \widehat{\gamma}_2 \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{32} \end{pmatrix} = (Z_2'Z_2)^{-1}Z_2'y_2 \\
&= \begin{pmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.0546 & -0.0032 & -0.0165 \\ -0.0032 & 0.1746 & -0.0921 \\ -0.0165 & -0.0921 & 0.1213 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.1930 \\ 0.3841 \\ 0.1975 \end{pmatrix}
\end{aligned}
$$

The inverses of these matrices were calculated using Matlab.

b)

For equation 1,

$$
\begin{aligned}
\widehat{\delta}_1^{2SLS} &= \begin{pmatrix} \widehat{\gamma}_1 \\ \widehat{\beta}_{11} \end{pmatrix} = (Z_1'P_xZ_1)^{-1}Z_1'P_xy_1 \\
&= \left[\begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix}' \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix} \right]^{-1} \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{pmatrix}' \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 0.3688 \\ 0.5787 \end{pmatrix}
\end{aligned}
$$

For equation 2,

$$
\begin{aligned}
\widehat{\delta}_2^{2SLS} &= \begin{pmatrix} \widehat{\gamma}_2 \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{32} \end{pmatrix} = (Z_2'P_xZ_2)^{-1}Z_2'P_xy_2 \\
&= \begin{pmatrix} 0.484375 \\ 0.367188 \\ 0.109375 \end{pmatrix}
\end{aligned}
$$

c)
The reduced form coefficient matrix estimated by OLS is

$$
\begin{aligned}
\widehat{\Pi} &= (X'X)^{-1}X'Y \\
&= \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 3 \\ 3 & 6 \\ 5 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.2287 & -0.0160 & -0.0372 \\ -0.0160 & 0.1755 & -0.0904 \\ -0.0372 & -0.0904 & 0.1223 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 6 \\ 5 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.6809 & 0.3298 \\ 0.0106 & 0.3724 \\ 0.1915 & 0.2021 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\Pi} &= (X'X)^{-1}X'Y \\
&= \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 3 \\ 3 & 6 \\ 5 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.2287 & -0.0160 & -0.0372 \\ -0.0160 & 0.1755 & -0.0904 \\ -0.0372 & -0.0904 & 0.1223 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 6 \\ 5 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 0.6809 & 0.3298 \\ 0.0106 & 0.3724 \\ 0.1915 & 0.2021 \end{pmatrix}
\end{aligned}
$$

Indirectly,

$$
\begin{aligned}
\widehat{\Pi} &= \begin{pmatrix} \beta_{11} & 0 \\ 0 & \beta_{22} \\ 0 & \beta_{32} \end{pmatrix} \begin{pmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0.5787 & 0 \\ 0 & 0.3672 \\ 0 & 0.1094 \end{pmatrix} \begin{pmatrix} 1 & -0.484375 \\ -0.3688 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0.7046 & 0.3413 \\ 0.1049 & 0.4471 \\ 0.0491 & 0.1334 \end{pmatrix}
\end{aligned}
$$

Note that the first equation is overidentified and the second is exactly identified.

2)

a) Using the false structure approach, we have the following restrictions:

$$\Gamma F = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

The restrictions here are :

$$f_{12} + \gamma f_{22} = \gamma f_{11} + f_{21}$$
$$f_{11} + \gamma f_{21} = 1$$
$$\gamma f_{12} + f_{22} = 1$$

$$Also, \qquad BF = \begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \\ 0 & \beta_{23} \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

The only restriction here is $\beta_{23} f_{21} = 0$

Hence, $f_{21} = 0$, $f_{11} = 1$, $f_{12} + \gamma f_{22} = \gamma$ and $\gamma f_{12} + f_{22} = 1$ which imply $f_{12} = 0$ *and* $f_{22} = 1$.

Therefore, the only admissible false structure F is the identity matrix and thus, the model is identified. Indeed, it is exactly identified since $\beta_{13} = 0$ identifies the first equation (by order and rank conditions) and the other restriction allows to identifiy the second equation.

b) Using the false structure approach, we have the following restrictions:

$$\Gamma F = \begin{pmatrix} 1 & 0 \\ \gamma_1 & 1 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

The restrictions here are :

$$f_{11} = 1$$
$$f_{12} = 0$$
$$\gamma_1 f_{12} + f_{22} = 1 \Longrightarrow f_{22} = 1$$

$$Also, \qquad F'\Sigma F = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

The restrictions here are both $f_{11} f_{12} \sigma_1^2 + f_{21} f_{22} \sigma_2^2 = 0$

Because $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$ and $f_{11} = f_{22} = 1$ and $f_{12} = 0$ and , it must be the case that $f_{21} f_{22} = 0$, and hence $f_{21} = 0$. *Therefore*, this model is also identified. Indeed, it is exactly identified. To see this, note that $\gamma_2 = 0$ exactly identifies the second equation and the other restriction allows the first equation to be also identified.