

Cornell University
Department of Economics

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Solution to Problem set # 7

1)

a) Let's figure out the probability limit of $\hat{\delta}$.

$$\hat{\delta} = \frac{\sum Y_{t-1}Y_t}{\sum Y_{t-1}^2} = \frac{\sum Y_{t-1}(\delta Y_{t-1} + u_t)}{\sum Y_{t-1}^2} = \delta + \frac{\sum Y_{t-1}u_t}{\sum Y_{t-1}^2}$$

By the WLLN, it follows

$$plim \hat{\delta} = \delta + \frac{plim \frac{1}{T} \sum Y_{t-1}u_t}{plim \frac{1}{T} \sum Y_{t-1}^2} = \delta + \frac{E(Y_{t-1}u_t)}{E(Y_{t-1}^2)}$$

Note that we are using WLLN in both, numerator and denominator, so we should check indeed that WLLN applies in both cases. That is, we should check that the random variables in question are uncorrelated, among other things. We are not going to do it, but try it. Note that the independence assumption here is crucial, since just ε_t white noise is not enough.

We will use the fact that both Y_t and u_t are stationary processes (why is Y_t stationary?).

$$\begin{aligned} E(Y_{t-1}u_t) &= E[(\delta Y_{t-2} + u_{t-1})u_t] = \delta E(Y_{t-2}u_t) + E(u_{t-1}u_t) \\ &= \delta E[(\delta Y_{t-3} + u_{t-2})u_t] + E(u_{t-1}u_t) \\ &= \delta^2 E[Y_{t-3}u_t] + \delta E(u_{t-2}u_t) + E(u_{t-1}u_t) \\ &= \delta^3 E[Y_{t-4}u_t] + \delta^2 E(u_{t-3}u_t) + \delta E(u_{t-2}u_t) + E(u_{t-1}u_t) \\ &= \sum_{j=0}^{\infty} \delta^j E(u_{t-j-1}u_t) \end{aligned}$$

Note that u_t is a stationary AR(1) process whose covariance function (see lecture 13 for derivation) is given by

$$E(u_{t-j-1}u_t) = \sigma_\varepsilon^2 \frac{\rho^{j+1}}{1-\rho^2}$$

Hence,

$$E(Y_{t-1}u_t) = \sum_{j=0}^{\infty} \delta^j \sigma_{\varepsilon}^2 \frac{\rho^{j+1}}{1-\rho^2} = \frac{\rho \sigma_{\varepsilon}^2}{1-\rho^2} \sum_{j=0}^{\infty} (\delta\rho)^j = \frac{\rho \sigma_{\varepsilon}^2}{1-\rho^2} \frac{1}{1-\delta\rho}$$

On the other hand;

$$\begin{aligned} E(Y_{t-1}^2) &= E(\delta^2 Y_{t-2}^2 + 2\delta Y_{t-2} u_{t-1} + u_{t-1}^2) \\ &= \delta^2 E(Y_{t-2}^2) + 2\delta E(Y_{t-2} u_{t-1}) + E(u_{t-1}^2) \\ &= \delta^2 E(Y_{t-1}^2) + 2\delta E(Y_{t-2} u_{t-1}) + E(u_{t-1}^2) \text{ since } Y_t \text{ is stationary} \\ &= \delta^2 E(Y_{t-1}^2) + 2\delta E(Y_{t-2} u_{t-1}) + \text{Var}(u_{t-1}) \text{ since } E(u_{t-1}) = 0 \\ &= \delta^2 E(Y_{t-1}^2) + 2\delta E(Y_{t-2} u_{t-1}) + \frac{\sigma_{\varepsilon}^2}{1-\rho^2} \text{ since } u_t \text{ is an AR(1)} \end{aligned}$$

Therefore, since stationarity of Y_t implies $E(Y_{t-2} u_{t-1}) = E(Y_{t-1} u_t)$ and $E(Y_{t-1}^2) = E(Y_{t-2}^2)$, we have

$$\begin{aligned} (1-\delta^2) E(Y_{t-1}^2) &= 2\delta \frac{\sigma_{\varepsilon}^2 \rho}{1-\rho^2} \frac{1}{1-\delta\rho} + \frac{\sigma_{\varepsilon}^2}{1-\rho^2} \\ &= \frac{\sigma_{\varepsilon}^2}{1-\rho^2} \left[1 + \frac{2\delta\rho}{1-\delta\rho} \right] \\ E(Y_{t-1}^2) &= \frac{\sigma_{\varepsilon}^2}{(1-\rho^2)(1-\delta^2)} \left[1 + \frac{2\delta\rho}{1-\delta\rho} \right] \\ &= \frac{\sigma_{\varepsilon}^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1-\delta\rho}{1-\delta\rho} + \frac{2\delta\rho}{1-\delta\rho} \right] \\ &= \frac{\sigma_{\varepsilon}^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right] \end{aligned}$$

Then,

$$\begin{aligned} \text{plim} \widehat{\delta} &= \delta + \frac{\frac{\rho \sigma_{\varepsilon}^2}{1-\rho^2} \frac{1}{1-\delta\rho}}{\frac{\sigma_{\varepsilon}^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right]} \\ &= \delta + \frac{\rho(1-\delta^2)}{(1+\delta\rho)} \neq \delta \quad \text{unless } \rho = 0 \end{aligned}$$

b)

$$\widehat{\rho} = \frac{\sum \widehat{u}_t \widehat{u}_{t-1}}{\sum \widehat{u}_{t-1}^2}$$

Hence,

$$\begin{aligned}\text{plim} \hat{\rho} &= \frac{\text{plim} \frac{1}{T} \sum \hat{u}_t \hat{u}_{t-1}}{\text{plim} \frac{1}{T} \sum \hat{u}_{t-1}^2} \\ &= \frac{\text{plim} \frac{1}{T} \sum \hat{u}_t \hat{u}_{t-1}}{\text{plim} \frac{1}{T} \sum \hat{u}_t^2}\end{aligned}$$

Since

$$\begin{aligned}\text{plim} \frac{1}{T} \sum \hat{u}_t^2 &= \text{plim} \frac{1}{T} \sum (Y_t - \hat{\delta} Y_{t-1})^2 \\ &= \text{plim} \frac{1}{T} \sum Y_t^2 - 2\text{plim} \hat{\delta} \text{plim} \frac{1}{T} \sum Y_t Y_{t-1} + \text{plim} \hat{\delta}^2 \text{plim} \frac{1}{T} \sum Y_{t-1}^2\end{aligned}$$

and

$$\begin{aligned}\text{plim} \frac{1}{T} \sum \hat{u}_t \hat{u}_{t-1} &= \text{plim} \frac{1}{T} \sum (Y_t - \hat{\delta} Y_{t-1})(Y_{t-1} - \hat{\delta} Y_{t-2}) \\ &= \text{plim} \frac{1}{T} \sum Y_t Y_{t-1} - \text{plim} \hat{\delta} \text{plim} \frac{1}{T} \sum Y_t Y_{t-2} \\ &\quad - \text{plim} \hat{\delta} \text{plim} \frac{1}{T} \sum Y_{t-1}^2 + \text{plim} \hat{\delta}^2 \text{plim} \frac{1}{T} \sum Y_{t-1} Y_{t-2}\end{aligned}$$

By a proper version of WLLN (make sure that you understand why WLLN applies here again), we have

$$\begin{aligned}\text{plim} \frac{1}{T} \sum Y_t Y_{t-1} &= E(Y_t Y_{t-1}) \\ \text{plim} \frac{1}{T} \sum Y_t Y_{t-2} &= E(Y_t Y_{t-2}) \\ \text{plim} \frac{1}{T} \sum Y_{t-1}^2 &= E(Y_{t-1}^2) \\ \text{plim} \frac{1}{T} \sum Y_{t-1} Y_{t-2} &= E(Y_{t-1} Y_{t-2})\end{aligned}$$

Let's do the following calculation, that will come up later on:

$$\begin{aligned}E(Y_{t-2} u_t) &= E[(\delta Y_{t-3} + u_{t-2}) u_t] = \delta E(Y_{t-3} u_t) + E(u_{t-2} u_t) \\ &= \delta^2 E[Y_{t-4} u_t] + \delta E(u_{t-3} u_t) + E(u_{t-2} u_t) \\ &= \sum_{j=0}^{\infty} \delta^j E(u_{t-j-2} u_t) = \sum_{j=0}^{\infty} \delta^j \sigma_{\varepsilon}^2 \frac{\rho^{j+2}}{1-\rho^2} \\ &= \frac{\sigma_{\varepsilon}^2 \rho^2}{1-\rho^2} \sum_{j=0}^{\infty} \delta^j \rho^j = \frac{\sigma_{\varepsilon}^2 \rho^2}{1-\rho^2} \frac{1}{1-\delta\rho} \\ &= \rho E(Y_{t-1} u_t) \quad \text{using our previous calculation}\end{aligned}$$

Now,

$$\begin{aligned}
E(Y_{t-1}^2) &= \frac{\sigma_\varepsilon^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right] \text{ from our previous calculation;} \\
E(Y_t Y_{t-1}) &= E((\delta Y_{t-1} + u_t) Y_{t-1}) = \delta E(Y_{t-1}^2) + E(u_t Y_{t-1}) \\
&= \frac{\delta \sigma_\varepsilon^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right] + E(u_t Y_{t-1}) \\
&= \frac{\delta \sigma_\varepsilon^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right] + \frac{\rho \sigma_\varepsilon^2}{1-\rho^2} \frac{1}{1-\delta\rho} \\
&= \frac{\sigma_\varepsilon^2(\delta+\rho)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)}; \\
E(Y_t Y_{t-2}) &= E[(\delta Y_{t-1} + u_t) Y_{t-2}] = \delta E(Y_{t-1} Y_{t-2}) + E(u_t Y_{t-2}) \\
&= \delta E(Y_t Y_{t-1}) + E(u_t Y_{t-2}) \text{ by the stationarity of } Y_t \\
&= \delta \frac{\sigma_\varepsilon^2(\delta+\rho)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)} + \frac{\sigma_\varepsilon^2 \rho^2}{1-\rho^2} \frac{1}{1-\delta\rho} \\
&= \sigma_\varepsilon^2 \frac{\delta(\delta+\rho) + \rho^2(1-\delta^2)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)} \\
&= \sigma_\varepsilon^2 \frac{\rho^2 + \delta^2 + \delta\rho - \rho^2\delta^2}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)}; \\
E(Y_{t-1} Y_{t-2}) &= E(Y_t Y_{t-1}) \text{ by the stationarity of } Y_t
\end{aligned}$$

Collecting all probability limits, we have

$$\begin{aligned}
\text{plim} \frac{1}{T} \sum \hat{u}_t^2 &= E(Y_t^2) - 2 \text{plim} \hat{\delta} E(Y_t Y_{t-1}) + \text{plim} \hat{\delta}^2 E(Y_{t-1}^2) \\
&= (1 + \text{plim} \hat{\delta}^2) E(Y_t^2) - 2 \text{plim} \hat{\delta} E(Y_t Y_{t-1})
\end{aligned}$$

and

$$\begin{aligned}
\text{plim} \frac{1}{T} \sum \hat{u}_t \hat{u}_{t-1} &= E(Y_t Y_{t-1}) - \text{plim} \hat{\delta} E(Y_t Y_{t-2}) - \text{plim} \hat{\delta} E(Y_{t-1}^2) + \text{plim} \hat{\delta}^2 E(Y_{t-1} Y_{t-2}) \\
&= E(Y_t Y_{t-1}) - \text{plim} \hat{\delta} E(Y_t Y_{t-2}) - \text{plim} \hat{\delta} E(Y_{t-1}^2) + \text{plim} \hat{\delta}^2 E(Y_t Y_{t-1}) \\
&= [1 + \text{plim} \hat{\delta}^2] E(Y_t Y_{t-1}) - \text{plim} \hat{\delta} [E(Y_t Y_{t-2}) + E(Y_{t-1}^2)]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{plim } \hat{\rho} &= \frac{\text{plim} \frac{1}{T} \sum \hat{u}_t \hat{u}_{t-1}}{\text{plim} \frac{1}{T} \sum \hat{u}_t^2} \\
&= \frac{\left[1 + \text{plim} \hat{\delta}^2 \right] E(Y_t Y_{t-1}) - \text{plim} \hat{\delta} [E(Y_t Y_{t-2}) + E(Y_{t-1}^2)]}{\left(1 + \text{plim} \hat{\delta}^2 \right) E(Y_t^2) - 2 \text{plim} \hat{\delta} E(Y_t Y_{t-1})} \\
&= \frac{\left[1 + \left(\delta + \frac{\rho(1-\delta^2)}{(1+\delta\rho)} \right)^2 \right] E(Y_t Y_{t-1}) - \left(\delta + \frac{\rho(1-\delta^2)}{(1+\delta\rho)} \right) \cdot \left[\frac{\sigma_\varepsilon^2(\rho^2 + \delta^2 + \delta\rho - \rho^2\delta^2)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)} + \frac{\sigma_\varepsilon^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right] \right]}{\left[1 + \left(\delta + \frac{\rho(1-\delta^2)}{(1+\delta\rho)} \right)^2 \right] \frac{\sigma_\varepsilon^2}{(1-\rho^2)(1-\delta^2)} \left[\frac{1+\delta\rho}{1-\delta\rho} \right] - 2 \left(\delta + \frac{\rho(1-\delta^2)}{(1+\delta\rho)} \right) E(Y_t Y_{t-1})} \\
&= \frac{\left[1 + \left(\frac{\delta+\rho}{1+\delta\rho} \right)^2 \right] \frac{\sigma_\varepsilon^2(\delta+\rho)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)} - \frac{\delta+\rho}{1+\delta\rho} \frac{\sigma_\varepsilon^2(1+2\delta\rho+\rho^2+\delta^2-\delta^2\rho^2)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)}}{\left[1 + \left(\frac{\delta+\rho}{1+\delta\rho} \right)^2 \right] \frac{\sigma_\varepsilon^2(1+\delta\rho)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)} - 2 \frac{\delta+\rho}{1+\delta\rho} \frac{\sigma_\varepsilon^2(\delta+\rho)}{(1-\rho^2)(1-\delta^2)(1-\delta\rho)}} \\
&= \frac{\left[1 + \left(\frac{\delta+\rho}{1+\delta\rho} \right)^2 \right] (\delta+\rho) - \frac{\delta+\rho}{1+\delta\rho}(1+2\delta\rho+\rho^2+\delta^2-\delta^2\rho^2)}{\left[1 + \left(\frac{\delta+\rho}{1+\delta\rho} \right)^2 \right] (1+\delta\rho) - 2 \frac{\delta+\rho}{1+\delta\rho} (\delta+\rho)} \\
&= \frac{\left[\frac{(1+\delta\rho)^2+(\delta+\rho)^2}{(1+\delta\rho)^2} \right] (\delta+\rho) - \frac{\delta+\rho}{(1+\delta\rho)^2} (1+\delta\rho)(1+2\delta\rho+\rho^2+\delta^2-\delta^2\rho^2)}{\left[\frac{(1+\delta\rho)^2+(\delta+\rho)^2}{(1+\delta\rho)^2} \right] (1+\delta\rho) - 2 \frac{\delta+\rho}{(1+\delta\rho)^2} (\delta+\rho)(1+\delta\rho)} \\
&= \frac{(\delta+\rho) \{ [(1+\delta\rho)^2 + (\delta+\rho)^2] - (1+\delta\rho)(1+2\delta\rho+\rho^2+\delta^2-\delta^2\rho^2) \}}{(1+\delta\rho) \{ [(1+\delta\rho)^2 + (\delta+\rho)^2] - 2(\delta+\rho)^2 \}} \\
&= \frac{(\delta+\rho)}{(1+\delta\rho)} \frac{\{ [(1+\delta\rho)^2 + (\delta+\rho)^2] - (1+\delta\rho)(1+2\delta\rho+\rho^2+\delta^2-\delta^2\rho^2) \}}{(1+\delta\rho)^2 - (\delta+\rho)^2} \\
&= \frac{\delta\rho(\delta+\rho)}{1+\delta\rho}
\end{aligned}$$

By observing that $\text{plim } \hat{\rho} = \rho$ implies $\delta = \pm 1$, which violates the assumption of the exercise, we show that $\hat{\rho}$ is inconsistent.

c)

For the Durbin-Watson statistic;

$$\begin{aligned}
d &= \frac{\sum (\hat{u}_t - \hat{u}_{t-1})^2}{\sum \hat{u}_t^2} = \frac{\sum \hat{u}_t^2 - 2 \sum \hat{u}_t \hat{u}_{t-1} + \sum \hat{u}_{t-1}^2}{\sum \hat{u}_t^2} \\
&= 2 - 2 \frac{\sum \hat{u}_t \hat{u}_{t-1}}{\sum \hat{u}_t^2} = 2 - 2\hat{\rho} \quad \text{for large enough T}
\end{aligned}$$

Therefore,

$$\begin{aligned}\text{plim}_d d &= 2 - 2\text{plim} \hat{\rho} \\ &= 2 \left(1 - \frac{\delta\rho(\delta + \rho)}{1 + \delta\rho} \right)\end{aligned}$$

2)

a) This process is covariance stationary:

$$\begin{aligned}E(y_t) &= E[z_1 \cos(ct) + z_2 \sin(ct)] \\ &= E(z_1) \cos(ct) + E(z_2) \sin(ct) \\ &= 0 \quad \forall t\end{aligned}$$

$$\begin{aligned}\text{Cov}(y_t, y_{t-j}) &= E(y_t y_{t-j}) \quad \text{because } E(y_t) = 0 \quad \forall t \\ &= E\{[z_1 \cos(ct) + z_2 \sin(ct)][z_1 \cos(ct - cj) + z_2 \sin(ct - cj)]\} \\ &= E\{[z_1 \cos(ct) + z_2 \sin(ct)] \\ &\quad [z_1(\cos(ct) \cos(cj) + \sin(ct) \sin(cj)) + z_2(\sin(ct) \cos(cj) - \cos(ct) \sin(cj))]\} \\ &= \sigma^2 \{ \cos(ct)[\cos(ct) \cos(cj) + \sin(ct) \sin(cj)] + \\ &\quad + \sin(ct)[\sin(ct) \cos(cj) - \cos(ct) \sin(cj)] \} \text{ by independence of the } z's \\ &= \sigma^2 \{ \cos(ct) \cos(ct) \cos(cj) + \sin(ct) \sin(ct) \cos(cj) \} \\ &= \sigma^2 \cos(cj)\end{aligned}$$

b) This process is also covariance stationary.

$$E(y_t) = E(z_t z_{t-1}) = E(z_t) E(z_{t-1}) = 0 \quad \forall t$$

$$\begin{aligned}\text{Cov}(y_t, y_{t-j}) &= E(y_t y_{t-j}) \quad \text{because } E(y_t) = 0 \quad \forall t \\ &= E\{[z_t z_{t-1}][z_{t-j} z_{t-j-1}]\} \\ &= \sigma^4 \text{ if } j = 0 \\ &= 0 \quad \text{if } j \neq 0\end{aligned}$$

c) This process is not covariance stationary.

$$\begin{aligned}E(y_t) &= E[z_t \cos(ct) + z_{t-1} \sin(ct)] \\ &= E(z_t) \cos(ct) + E(z_{t-1}) \sin(ct) \\ &= 0 \quad \forall t\end{aligned}$$

$$\begin{aligned}
Cov(y_t, y_{t-j}) &= E(y_t y_{t-j}) \quad \text{because } E(y_t) = 0 \quad \forall t \\
&= E\{[z_t \cos(ct) + z_{t-1} \sin(ct)][z_{t-j} \cos(ct - cj) + z_{t-j-1} \sin(ct - cj)]\} \\
&= E\{[z_t \cos(ct) + z_{t-1} \sin(ct)] \\
&\quad [z_{t-j}(\cos(ct) \cos(cj) + \sin(ct) \sin(cj)) + z_{t-j-1}(\sin(ct) \cos(cj) - \cos(ct) \sin(cj))]\} \\
&= \sigma^2 \quad \text{if } j = 0 \\
&= \sigma^2 \sin(ct)[\cos(ct) \cos(c) + \sin(ct) \sin(c)] = \sigma^2 \sin(ct) \cos(c(t-1)) \quad \text{if } j = 1 \\
&= 0 \quad \text{if } |j| \geq 2
\end{aligned}$$

This process is not covariance stationary because when $j=1$, $Cov(y_t, y_{t-j})$ depends on t unless $c = 0$ (or, more generally, unless $c = \pi k$ or $c = \frac{(2k+1)}{2}\pi$, with k integer).