

Cornell University
Department of Economics

Econ 620

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Solution to Problem set # 6

1. The log likelihood function is

$$\begin{aligned}
 \log L(\alpha, \beta) &= \sum_{i=1}^N \log f(y_i | x_i; \alpha, \beta) = \sum_{i=1}^N \{\log \gamma_i + \log [\exp(-\gamma_i y_i)]\} \\
 &= \sum_{i=1}^N \{\log [\exp[\alpha + \beta x_i]] - \gamma_i y_i\} \\
 &= \sum_{i=1}^N [(\alpha + \beta x_i) - y_i \exp(\alpha + \beta x_i)]
 \end{aligned}$$

Differentiate with respect to α and β ;

$$\begin{aligned}
 \frac{\partial \log L}{\partial \alpha} &= \sum_{i=1}^N [1 - y_i \exp(\alpha + \beta x_i)] \\
 \frac{\partial \log L}{\partial \beta} &= \sum_{i=1}^N [x_i - x_i y_i \exp(\alpha + \beta x_i)]
 \end{aligned}$$

Setting the first order conditions equal to zero and solving them will give us the MLE. - The analytic solution is not available. You have to solve them numerically-. To get the information matrix,

$$\begin{aligned}
 \frac{\partial^2 \log L}{\partial \alpha^2} &= - \sum_{i=1}^N y_i \exp(\alpha + \beta x_i) \\
 \frac{\partial^2 \log L}{\partial \alpha \partial \beta} &= - \sum_{i=1}^N x_i y_i \exp(\alpha + \beta x_i) \\
 \frac{\partial^2 \log L}{\partial \beta^2} &= - \sum_{i=1}^N x_i^2 y_i \exp(\alpha + \beta x_i)
 \end{aligned}$$

Assigning minus sign and taking expectations;

$$\begin{aligned} E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) &= \sum_{i=1}^N E(y_i) \exp(\alpha + \beta x_i) \\ E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) &= \sum_{i=1}^N E(y_i) x_i \exp(\alpha + \beta x_i) \\ E\left(\frac{\partial^2 \log L}{\partial \beta^2}\right) &= \sum_{i=1}^N E(y_i) x_i^2 \exp(\alpha + \beta x_i) \end{aligned}$$

However,

$$E(y) = \frac{1}{\gamma_i} = \frac{1}{\exp(\alpha + \beta x_i)}$$

Then, the information matrix is given by

$$i(\alpha, \beta) = \begin{bmatrix} N & \sum_{i=1}^N x_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N x_i^2 \end{bmatrix}$$

Therefore, by letting $\bar{\alpha}$ and $\bar{\beta}$ be the ML estimators of α and β respectively, we have

$$\begin{bmatrix} \sqrt{N}(\bar{\alpha} - \alpha) \\ \sqrt{N}(\bar{\beta} - \beta) \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \frac{\sum_{i=1}^N x_i}{N} \\ \frac{\sum_{i=1}^N x_i}{N} & \frac{\sum_{i=1}^N x_i^2}{N} \end{bmatrix}^{-1}\right)$$

i.e.

$$\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} \stackrel{a}{\sim} N\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} N & \sum_{i=1}^N x_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N x_i^2 \end{bmatrix}^{-1}\right)$$

We also know that

$$\bar{\delta} \xrightarrow{d} N(\delta, q)$$

where

$$\begin{aligned} q &= \begin{bmatrix} \frac{\partial \delta}{\partial \alpha} & \frac{\partial \delta}{\partial \beta} \end{bmatrix} \begin{bmatrix} N & \sum_{i=1}^N x_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \delta}{\partial \alpha} \\ \frac{\partial \delta}{\partial \beta} \end{bmatrix} \\ &= [\exp(\beta) \quad \bar{\alpha} \exp(\beta)] N \begin{bmatrix} N & \sum_{i=1}^N x_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \exp(\beta) \\ \alpha \exp(\beta) \end{bmatrix} \end{aligned}$$

2.

(a)

$$\hat{\beta}_{OLS} = (X'X)^{-1} X'y = \beta + (X'X)^{-1} X'\varepsilon$$

and

$$E(\hat{\beta}_{OLS}) = \beta$$

Hence,

$$\begin{aligned} Var(\hat{\beta}_{OLS}) &= E\left[\left(\hat{\beta}_{OLS} - E(\hat{\beta}_{OLS})\right)\left(\hat{\beta}_{OLS} - E(\hat{\beta}_{OLS})\right)'\right] \\ &= E\left[(X'X)^{-1} X'\varepsilon\varepsilon' X (X'X)^{-1}\right] = \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1} \end{aligned}$$

On the other hand,

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}y = \beta + (X'\Omega^{-1}X)^{-1} X'\Omega^{-1}\varepsilon$$

and

$$E(\hat{\beta}_{GLS}) = \beta$$

Therefore,

$$\begin{aligned} Var(\hat{\beta}_{GLS}) &= E\left[\left(\hat{\beta}_{GLS} - E(\hat{\beta}_{GLS})\right)\left(\hat{\beta}_{GLS} - E(\hat{\beta}_{GLS})\right)'\right] \\ &= E\left[(X'\Omega^{-1}X)^{-1} X'\Omega^{-1}\varepsilon\varepsilon'\Omega^{-1} X (X'\Omega^{-1}X)^{-1}\right] = \sigma^2 (X'\Omega^{-1}X)^{-1} \end{aligned}$$

(b) First of all,

$$e = y - X\hat{\beta}_{OLS} = [I - X(X'X)^{-1}X']y = My = M\varepsilon$$

Then,

$$E(e) = ME(\varepsilon) = 0$$

Hence,

$$\begin{aligned} Var(e) &= E[(e - E(e))(e - E(e))'] = E[ee'] = E[M\varepsilon\varepsilon'M] = \sigma^2 M\Omega M \\ &= \sigma^2 [I - X(X'X)^{-1}X']\Omega[I - X(X'X)^{-1}X'] \\ &= \sigma^2 \left[\begin{array}{c} \Omega - \Omega X(X'X)^{-1}X' - X(X'X)^{-1}X'\Omega \\ + X(X'X)^{-1}X'\Omega X(X'X)^{-1}X' \end{array} \right] \end{aligned}$$

(c) By the same method,

$$\tilde{e} = y - X\hat{\beta}_{GLS} = [I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}]y = M_\Omega y = M_\Omega\varepsilon$$

since

$$M_\Omega y = M_\Omega(X\beta + \varepsilon) = M_\Omega X\beta + M_\Omega\varepsilon = M_\Omega\varepsilon$$

then,

$$E(\tilde{e}) = M_\Omega E(\varepsilon) = 0$$

Therefore,

$$\begin{aligned} Var(\tilde{e}) &= E[(\tilde{e} - E(\tilde{e}))(\tilde{e} - E(\tilde{e}))'] = E[\tilde{e}\tilde{e}'] = E[M_\Omega \varepsilon \varepsilon' M_\Omega'] \\ &= \sigma^2 M_\Omega \Omega M_\Omega' \\ &= \sigma^2 [I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}] \Omega [I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}]' \\ &= \sigma^2 \left[\Omega - \Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1}X' - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Omega \right. \\ &\quad \left. + X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1}X' \right] \\ &= \sigma^2 [\Omega - X(X'\Omega^{-1}X)^{-1}X' - X(X'\Omega^{-1}X)^{-1}X' + X(X'\Omega^{-1}X)^{-1}X'] \\ &= \sigma^2 [\Omega - X(X'\Omega^{-1}X)^{-1}X'] \\ &= \sigma^2 [I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}] \Omega = \sigma^2 M_\Omega \Omega \end{aligned}$$

(d) Note that

$$E(e) = 0 \text{ and } E(\tilde{e}) = 0$$

Then,

$$\begin{aligned} Cov(e, \tilde{e}) &= E[(e - E(e))(\tilde{e} - E(\tilde{e}))'] = E[e\tilde{e}'] = E[M\varepsilon\varepsilon' M_\Omega'] \\ &= \sigma^2 M_\Omega \Omega M_\Omega' = \sigma^2 [I - X(X'X)^{-1}X'] \Omega [I - X(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}]' \\ &= \sigma^2 \left[\Omega - \Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1}X' - X(X'X)^{-1}X'\Omega \right. \\ &\quad \left. + X(X'X)^{-1}X'\Omega\Omega^{-1}X(X'\Omega^{-1}X)^{-1}X' \right] \\ &= \sigma^2 [\Omega - X(X'\Omega^{-1}X)^{-1}X' - X(X'X)^{-1}X'\Omega + X(X'\Omega^{-1}X)^{-1}X'] \\ &= \sigma^2 [\Omega - X(X'X)^{-1}X'\Omega] = \sigma^2 M\Omega \end{aligned}$$

3. The model is specified as

$$\begin{aligned} Y_t &= \delta Y_{t-1} + u_t & |\delta| < 1 & (*) \\ u_t &= \varepsilon_t + \alpha \varepsilon_{t-1} & \text{with } \varepsilon \sim N(0, \sigma_\varepsilon^2) & (**) \end{aligned}$$

(a) The parameter δ is estimated by $\hat{\delta} = \frac{\sum Y_t Y_{t-1}}{\sum Y_{t-1}^2}$. Hence,

$$\hat{\delta} = \frac{\sum Y_{t-1} (\delta Y_{t-1} + u_t)}{\sum Y_{t-1}^2} = \delta + \frac{\sum Y_{t-1} u_t}{\sum Y_{t-1}^2}$$

Therefore,

$$\text{plim } \hat{\delta} = \delta + \frac{\text{plim } T^{-1} \sum Y_{t-1} u_t}{\text{plim } T^{-1} \sum Y_{t-1}^2}$$

If we can find $\text{plim } T^{-1} \sum Y_{t-1} u_t$ and $\text{plim } T^{-1} \sum Y_{t-1}^2$, we are done. Recursively substituting in $(*)$ gives

$$Y_t = \sum_{j=0}^{\infty} \delta^j u_{t-j} = \sum_{j=0}^{\infty} \delta^j (\varepsilon_{t-j} + \alpha \varepsilon_{t-j-1}) = \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j} + \alpha \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-1}$$

Therefore, we have

$$\begin{aligned} Y_{t-1} &= \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-1-j} - \alpha \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-2} \\ u_t &= \varepsilon_t + \alpha \varepsilon_{t-1} \end{aligned}$$

Multiplying two terms and taking expectation,

$$\begin{aligned} E(Y_{t-1} u_t) &= E \left[\left(\sum_{j=0}^{\infty} \delta^j \varepsilon_{t-1-j} + \alpha \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-2} \right) (\varepsilon_t + \alpha \varepsilon_{t-1}) \right] \\ &= E[\alpha \varepsilon_{t-1}^2] = \alpha \sigma_\varepsilon^2 \end{aligned}$$

On the other hand,

$$\begin{aligned} E(Y_{t-1}^2) &= E \left[\sum_{j=0}^{\infty} \delta^j \varepsilon_{t-1-j} \right]^2 + 2\alpha E \left[\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^j \varepsilon_{t-1-j} \delta^k \varepsilon_{t-k-2} \right] + \alpha^2 \left[\sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-2} \right]^2 \\ &= \sum_{j=0}^{\infty} \delta^{2j} E(\varepsilon_{t-1-j}^2) + 2\alpha \sum_{k=0}^{\infty} \delta^{2k+1} E(\varepsilon_{t-k-2}^2) + \alpha^2 \sum_{j=0}^{\infty} \delta^{2j} E(\varepsilon_{t-2-j}^2) \\ &= \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \delta^{2j} + 2\alpha \sigma_\varepsilon^2 \sum_{k=0}^{\infty} \delta^{2k+1} + \alpha^2 \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \delta^{2j} \\ &= \sigma_\varepsilon^2 \left[\frac{1}{1-\delta^2} + 2\alpha \frac{\delta}{1-\delta^2} + \alpha^2 \frac{1}{1-\delta^2} \right] \\ &= \frac{1+2\alpha\delta+\alpha^2}{1-\delta^2} \sigma_\varepsilon^2 \end{aligned}$$

In sum,

$$\begin{aligned}
\text{plim} \widehat{\delta} &= \delta + \frac{\text{plim} n^{-1} \sum Y_{t-1} u_t}{\text{plim} n^{-1} \sum Y_{t-1}^2} = \delta + \frac{E(Y_{t-1} u_t)}{E(Y_{t-1}^2)} \\
&= \delta + \frac{\alpha \sigma_\varepsilon^2}{\left[\frac{1+2\alpha\delta+\alpha^2}{1-\delta^2} \right] \sigma_\varepsilon^2} \\
&= \delta + \frac{\alpha(1-\delta^2)}{1+2\alpha\delta+\alpha^2} = \delta + \frac{\frac{1}{1+\alpha^2}\alpha(1-\delta^2)}{\frac{1}{1+\alpha^2}(1+2\alpha\delta+\alpha^2)} \\
&= \delta + \frac{\phi(1-\delta^2)}{1+\frac{\alpha}{1+\alpha^2}2\delta} = \delta + \frac{\phi(1-\delta^2)}{1+2\phi\delta}
\end{aligned}$$

Make sure that you understand why WLLN applies. You need to check that the conditions for WLLN are satisfied.

(b) To find $\text{plim} \frac{1}{T} \sum \widehat{u}_t^2$;

$$\frac{1}{T} \sum \widehat{u}_t^2 = \frac{1}{T} \sum (Y_t - \widehat{\delta} Y_{t-1})^2 = \frac{1}{T} \sum Y_t^2 - 2\widehat{\delta} \frac{1}{T} \sum Y_t Y_{t-1} + \widehat{\delta}^2 \frac{1}{T} \sum Y_{t-1}^2$$

By a proper version of WLLN, we have

$$\begin{aligned}
\text{plim} \frac{1}{T} \sum Y_t^2 &= E(Y_t^2) \\
\text{plim} \frac{1}{T} \sum Y_t Y_{t-1} &= E(Y_t Y_{t-1}) \\
\text{plim} \frac{1}{T} \sum Y_{t-1}^2 &= E(Y_{t-1}^2)
\end{aligned}$$

On the other hand, we know, from part (a), that

$$\begin{aligned}
\text{plim} \widehat{\delta} &= \delta + \frac{\phi(1-\delta^2)}{1-2\phi\delta} \equiv \delta + \alpha^* \\
\text{plim} \widehat{\delta}^2 &= (\text{plim} \widehat{\delta})^2 = (\delta + \alpha^*)^2
\end{aligned}$$

Again, from the recursive solution of Y_t ,

$$\begin{aligned}
E(Y_t^2) &= E \left(\sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j} + \alpha \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-1} \right)^2 \\
&= \sum_{j=0}^{\infty} \delta^{2j} E(\varepsilon_{t-j}^2) + \alpha^2 \sum_{j=0}^{\infty} \delta^{2j} E(\varepsilon_{t-j-1}^2) + 2\alpha \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^j \delta^k E(\varepsilon_{t-j} \varepsilon_{t-k-1}) \\
&= \sum_{j=0}^{\infty} \delta^{2j} E(\varepsilon_{t-j}^2) + \alpha^2 \sum_{j=0}^{\infty} \delta^{2j} E(\varepsilon_{t-j-1}^2) + 2\alpha \sum_{k=0}^{\infty} \delta^{2k+1} E(\varepsilon_{t-k-1}^2) \\
&= \frac{(1+\alpha^2) + 2\alpha\delta}{1-\delta^2} \sigma_\varepsilon^2 = E(Y_{t-1}^2)
\end{aligned}$$

And,

$$\begin{aligned}
E(Y_t Y_{t-1}) &= E \left[\left(\sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j} + \alpha \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-1} \right) \left(\sum_{j=0}^{\infty} \delta^j \varepsilon_{t-1-j} + \alpha \sum_{j=0}^{\infty} \delta^j \varepsilon_{t-j-2} \right) \right] \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^{j+k} E(\varepsilon_{t-j} \varepsilon_{t-k-1}) + \alpha \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^{j+k} E(\varepsilon_{t-j} \varepsilon_{t-k-2}) \\
&\quad + \alpha \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^{j+k} E(\varepsilon_{t-j-1}^2) + \alpha^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \delta^{j+k} E(\varepsilon_{t-j-1} \varepsilon_{t-k-2}) \\
&= \sum_{k=0}^{\infty} \delta^{2k+1} E(\varepsilon_{t-k-1}^2) + \alpha \sum_{k=0}^{\infty} \delta^{2k+2} E(\varepsilon_{t-k-2}^2) + \alpha \sum_{k=0}^{\infty} \delta^{2k} E(\varepsilon_{t-k-1}^2) + \alpha^2 \sum_{k=0}^{\infty} \delta^{2k+1} E(\varepsilon_{t-k-2}^2) \\
&= \sigma_{\varepsilon}^2 \left[\frac{\delta}{1-\delta^2} + \frac{\alpha \delta^2}{1-\delta^2} + \frac{\alpha}{1-\delta^2} + \frac{\delta \alpha^2}{1-\delta^2} \right] = \sigma_{\varepsilon}^2 \frac{(\alpha+\delta)(\alpha\delta+1)}{1-\delta^2}
\end{aligned}$$

In sum,

$$\begin{aligned}
\text{plim} \frac{1}{n} \sum \hat{u}_t^2 &= E(Y_t^2) - 2 \left[\text{plim} \hat{\delta} \right] [E(Y_t Y_{t-1})] + \left[\text{plim} \hat{\delta}^2 \right] E(Y_{t-1}^2) \\
&= E(Y_t^2) \left[1 + \left[\text{plim} \hat{\delta}^2 \right] \right] - 2 \left[\text{plim} \hat{\delta} \right] [E(Y_t Y_{t-1})] \text{ since } E(Y_t^2) = E(Y_{t-1}^2) \\
&= \left[\frac{1+\alpha^2+2\alpha\delta}{1-\delta^2} \sigma_{\varepsilon}^2 \right] \left[1 + (\delta+\alpha^*)^2 \right] - 2[(\delta+\alpha^*)] \left[\sigma_{\varepsilon}^2 \frac{(\alpha+\delta)(\alpha\delta+1)}{1-\delta^2} \right] \\
&= \frac{\sigma_{\varepsilon}^2}{1-\delta^2} \left[(1+\alpha^2+2\alpha\delta) (1+(\delta+\alpha^*)^2) - 2(\delta+\alpha^*)(\alpha+\delta)(\alpha\delta+1) \right] \\
&= \sigma_{\varepsilon}^2 [1 + \alpha(\alpha - \alpha^*)]
\end{aligned}$$

4.

$$y_t = \frac{3L}{1-0.9L+0.2L^2} x_t + u_t$$

(a) Denote $A(L) = 3L$, $B(L) = 1 - 0.9L + 0.2L^2$ and $D(L) = \frac{A(L)}{B(L)}$.

Then, the total multiplier $= D(1) = \frac{A(1)}{B(1)} = \frac{3}{1-0.9+0.2} = 10$.

(b) the mean lag $= D'(1) = \frac{A'(1)}{A(1)} - \frac{B'(1)}{B(1)} = \frac{3}{3} - \frac{-0.9+0.4}{0.3} = \frac{8}{3}$.

(c)

$$\delta_0 = 0, \delta_1 = 3, \delta_2 = 2.7, \delta_3 = 1.83$$