

Cornell University
Department of Economics

Econ 620 - Spring 2004

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Solution to Problem set # 5

1. Autocorrelation function is defined as

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

where $\gamma(h) = Cov(X_t, X_{t-h}) = E[(X_t - E(X_t))(X_{t-h} - E(X_{t-h}))]$ for $h = 0, 1, 2, \dots$.

(a) This is an AR(1) process, and it was shown in lecture 13 that

$$\begin{aligned}\gamma(h) &= \sigma_\varepsilon^2 \frac{\alpha^h}{1-\alpha^2} \text{ for } h = 1, 2, \dots \\ \gamma(0) &= \sigma_\varepsilon^2 \frac{1}{1-\alpha^2}\end{aligned}$$

Hence,

$$\rho(h) = \begin{cases} 1 & \text{when } h = 0 \\ \alpha^h & \text{when } h = 1, 2, \dots \end{cases}$$

(b) This is an MA(2) process. From lecture 13 we have,

$$\begin{aligned}\gamma(h) &= \begin{cases} \sigma_\varepsilon^2(\theta_1 + \theta_2\theta_1) & \text{when } h = 1 \\ \sigma_\varepsilon^2\theta_2 & \text{when } h = 2 \\ 0 & \text{when } h \geq 3 \end{cases} \\ \gamma(0) &= \sigma_\varepsilon^2 [1 + \theta_1^2 + \theta_2^2]\end{aligned}$$

Hence,

$$\rho(h) = \begin{cases} \frac{\theta_1 + \theta_2\theta_1}{1+\theta_1^2+\theta_2^2} & \text{when } h = 1 \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{when } h = 2 \\ 0 & \text{when } h \geq 3 \end{cases}$$

(c) Let's calculate the variance of this ARMA(1,1) process. Note that the mean of the process is 0. Then,

$$\begin{aligned}\gamma(0) &= Var(Z_t) = E(Z_t^2) = E[(\rho Z_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1})^2] \\ &= E[\rho^2 Z_{t-1}^2 + \varepsilon_t^2 + \theta^2 \varepsilon_{t-1}^2 + 2\rho Z_{t-1} \varepsilon_t + 2\rho \theta Z_{t-1} \varepsilon_{t-1} + 2\theta \varepsilon_t \varepsilon_{t-1}] \\ &= \rho^2 E(Z_{t-1}^2) + \sigma_\varepsilon^2 + \theta^2 \sigma_\varepsilon^2 + 2\rho \theta E(Z_{t-1} \varepsilon_{t-1}) \\ &= \rho^2 \gamma(0) + \sigma_\varepsilon^2 + \theta^2 \sigma_\varepsilon^2 + 2\rho \theta E[\varepsilon_{t-1} (\rho Z_{t-2} + \varepsilon_{t-1} + \theta \varepsilon_{t-2})] \\ &= \rho^2 \gamma(0) + \sigma_\varepsilon^2 + \theta^2 \sigma_\varepsilon^2 + 2\rho \theta \sigma_\varepsilon^2\end{aligned}$$

Therefore,

$$\gamma(0) = \frac{\sigma_\varepsilon^2 [1 + 2\rho\theta + \theta^2]}{1 - \rho^2}$$

Note that the process is stationary. We will use the fact that $Cov(X_t, X_{t-h}) = Cov(X_s, X_{s-h})$ for all s, t , and h since the process is stationary.

$$\begin{aligned} E(Z_t Z_{t-1}) &= \rho E(Z_{t-1} Z_{t-1}) + E(\varepsilon_t Z_{t-1}) + E(\theta \varepsilon_{t-1} Z_{t-1}) \\ \gamma(1) &= \rho \gamma(0) + \theta \sigma_\varepsilon^2 \end{aligned}$$

since

$$\begin{aligned} E(\varepsilon_t Z_{t-1}) &= 0 \\ E(\theta \varepsilon_{t-1} Z_{t-1}) &= \theta E[\varepsilon_{t-1} (\rho Z_{t-2} + \varepsilon_{t-1} + \theta \varepsilon_{t-2})] \\ &= \theta \sigma_\varepsilon^2 \end{aligned}$$

Hence,

$$\gamma(1) = \rho \left[\frac{\sigma_\varepsilon^2 [1 + 2\rho\theta + \theta^2]}{1 - \rho^2} \right] + \theta \sigma_\varepsilon^2$$

For $h \geq 2$, we have

$$\begin{aligned} E(Z_t Z_{t-h}) &= \rho E(Z_{t-1} Z_{t-h}) + E(\varepsilon_t Z_{t-h}) + E(\theta \varepsilon_{t-1} Z_{t-h}) \\ \gamma(h) &= \rho \gamma(h-1) \end{aligned}$$

since

$$\begin{aligned} E(\varepsilon_t Z_{t-h}) &= 0 \\ E(\theta \varepsilon_{t-1} Z_{t-h}) &= 0 \text{ when } h \geq 2 \end{aligned}$$

Solving the first order homogenous difference equation, we have

$$\gamma(h) = \rho^{h-1} \gamma(1) \text{ for } h \geq 2$$

Therefore, we have

$$\rho(h) = \begin{cases} \rho + \frac{\theta}{1 + 2\rho\theta + \theta^2} & \text{when } h = 1 \\ \rho^{h-1} \left\{ \rho + \frac{\theta}{1 + 2\rho\theta + \theta^2} \right\} & \text{when } h \geq 2 \end{cases}$$

2. (a) The OLS estimator is given by

$$\hat{\beta} = \beta + (Z'Z)^{-1} Z'u$$

where

$$Z_t = \begin{bmatrix} 1 & X_t & Y_{t-1} \end{bmatrix}$$

Then,

$$\begin{aligned}\text{plim} \hat{\beta} &= \beta + \left(\text{plim} \frac{Z'Z}{T} \right)^{-1} \text{plim} \left(\frac{Z'u}{T} \right) \\ &= \beta + Q^{-1} \text{plim} \left(\frac{Z'u}{T} \right)\end{aligned}$$

where $Q = \text{plim} \frac{Z'Z}{T}$. Note that

$$\text{plim} \left(\frac{Z'u}{T} \right) = \text{plim} \left[\begin{array}{c} \frac{1}{T} \sum_{t=1}^T u_t \\ \frac{1}{T} \sum_{t=1}^T X_t u_t \\ \frac{1}{T} \sum_{t=1}^T Y_{t-1} u_t \end{array} \right]$$

We now apply a proper version of WLLN to conclude that

$$\begin{aligned}\text{plim} \frac{1}{T} \sum_{t=1}^n u_t &= E(u_t) = 0 \\ \text{plim} \frac{1}{T} \sum_{t=1}^n X_t u_t &= E(X_t u_t) = X_t E(u_t) = 0 \\ \text{plim} \frac{1}{T} \sum_{t=1}^n Y_{t-1} u_t &= E(Y_{t-1} u_t) = E[(\beta_1 + \beta_2 X_{t-1} + \beta_3 Y_{t-2} + u_{t-1}) u_t] \\ &= \beta_1 E(u_t) + \beta_2 X_{t-1} E(u_t) + \beta_3 E(Y_{t-2} u_t) + E(u_{t-1} u_t) \\ &= \beta_3 E((\beta_1 + \beta_2 X_{t-2} + \beta_3 Y_{t-3} + u_{t-2}) u_t) + E(u_{t-1} u_t) \\ &= \beta_3^2 E(Y_{t-3} u_t) + \beta_3 E(u_{t-2} u_t) + E(u_{t-1} u_t) \\ &= \beta_3^3 E(Y_{t-4} u_t) + \beta_3^2 E(u_{t-3} u_t) + \beta_3 E(u_{t-2} u_t) + E(u_{t-1} u_t) \\ &= \sum_{j=1}^{\infty} \beta_3^{j-1} E(u_{t-j} u_t)\end{aligned}$$

Now, recall that

$$E(u_{t-j} u_t) = \sigma_{\varepsilon}^2 \frac{\rho^j}{1 - \rho^2}$$

Hence,

$$\begin{aligned}\text{plim} \frac{1}{T} \sum_{t=1}^n Y_{t-1} u_t &= \sum_{j=1}^{\infty} \beta_3^{j-1} \sigma_{\varepsilon}^2 \frac{\rho^j}{1 - \rho^2} = \frac{\rho \sigma_{\varepsilon}^2}{1 - \rho^2} \sum_{j=1}^{\infty} (\rho \beta_3)^{j-1} \\ &= \frac{\rho \sigma_{\varepsilon}^2}{1 - \rho^2} \frac{1}{1 - \rho \beta_3},\end{aligned}$$

as long as $|\rho \beta_3| < 1$. Therefore,

$$\text{plim} \hat{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + Q^{-1} \begin{bmatrix} 0 \\ 0 \\ \frac{\sigma_{\varepsilon}^2 \rho}{(1 - \rho^2)(1 - \rho \beta_3)} \end{bmatrix} \neq \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The presence of lagged dependent variable on the right hand side makes **all coefficients** inconsistent when the error terms show some serial correlation.

(b) How do we cure the problem? The rescue comes from the instrumental variables estimator(IV). If we can find a variable w_t which is closely correlated with Y_{t-1} but not with u_{t-1} , we can obtain a consistent estimator of β through

$$\hat{\beta}_{IV} = (W'Z)^{-1} W'y$$

where

$$W_t = [\begin{array}{ccc} 1 & X_t & w_t \end{array}]$$

IV estimator is consistent since

$$\hat{\beta}_{IV} = \beta + (W'Z)^{-1} W'u$$

Then,

$$\begin{aligned} \text{plim} \hat{\beta}_{IV} &= \beta + \left(\text{plim} \frac{W'Z}{T} \right)^{-1} \left(\text{plim} \frac{W'u}{T} \right) \\ &= \beta + \Pi^{-1}\mathbf{0} = \beta \end{aligned}$$

where $\text{plim} \frac{W'Z}{T} = \Pi$ and $\text{plim} \frac{W'u}{T} = 0$ by a proper choice of instrument.

3. a)

$$s^2 = \frac{e'e}{N-k} = \frac{\varepsilon'M\varepsilon}{N-k}$$

where $M = [I - X(X'X)^{-1}X']$. Then, $E(s^2) = E\left(\frac{\varepsilon'M\varepsilon}{N-k}\right) = \frac{1}{N-k}E(\varepsilon'M\varepsilon)$.

Now, consider

$$\begin{aligned} E(\varepsilon'M\varepsilon) &= E[tr(\varepsilon'M\varepsilon)] \text{ since } \varepsilon'M\varepsilon \text{ is a scalar} \\ &= E[tr(M\varepsilon\varepsilon')] \text{ since } tr(AB) = tr(BA) \\ &= tr[ME(\varepsilon\varepsilon')] \\ &= tr[\sigma^2 M\Omega] \text{ since } E(\varepsilon\varepsilon') = \sigma^2\Omega \\ &= \sigma^2 tr\left([I - X(X'X)^{-1}X']\Omega\right) \\ &= \sigma^2 \left[tr(\Omega) - tr\left(X(X'X)^{-1}X'\Omega\right)\right] \\ &= \sigma^2 tr(\Omega) - \sigma^2 tr\left((X'X)^{-1}X'\Omega X\right) \\ &= \sigma^2 N - \sigma^2 tr\left((X'X)^{-1}X'\Omega X\right) \end{aligned}$$

Hence,

$$\begin{aligned} E(s^2) &= \frac{\sigma^2 N}{N-k} - \frac{\sigma^2 \text{tr} \left((X'X)^{-1} X' \Omega X \right)}{N-k} \\ &= \frac{\sigma^2 N}{N-k} - \frac{\sigma^2 \text{tr} \left[\left(\frac{X'X}{N-k} \right)^{-1} \left(\frac{X' \Omega X}{N-k} \right) \right]}{N-k} \end{aligned}$$

(b) It is straight forward to get

$$\begin{aligned} \lim_{N \rightarrow \infty} E(s^2) &= \lim_{N \rightarrow \infty} \frac{\sigma^2 N}{N-k} - \lim_{N \rightarrow \infty} \left(\frac{\sigma^2 \text{tr} \left[\left(\frac{X'X}{N-k} \right)^{-1} \left(\frac{X' \Omega X}{N-k} \right) \right]}{N-k} \right) \\ &= \sigma^2 - \frac{\sigma^2 \text{tr} [Q^{-1} L]}{\lim_{N \rightarrow \infty} (N-k)} = \sigma^2 \end{aligned}$$

(c) First of all, we will expand s^2 as;

$$s^2 = \frac{e'e}{N-k} = \frac{\varepsilon'M\varepsilon}{N-k} = \frac{\varepsilon'\varepsilon}{N-k} - \frac{\varepsilon'X(X'X)^{-1}X'\varepsilon}{N-k} = \frac{N}{N-k} \left[\frac{\varepsilon'\varepsilon}{N} - \frac{\varepsilon'X(X'X)^{-1}X'\varepsilon}{N} \right]$$

Then,

$$\begin{aligned} \text{plim } s^2 &= \left(\text{plim} \frac{N}{N-k} \right) \left(\text{plim} \left[\frac{\varepsilon'\varepsilon}{N} - \frac{\varepsilon'X(X'X)^{-1}X'\varepsilon}{N} \right] \right) \\ &= \text{plim} \frac{\varepsilon'\varepsilon}{N} - \text{plim} \frac{\varepsilon'X(X'X)^{-1}X'\varepsilon}{N} \\ &= \text{plim} \frac{\varepsilon'\varepsilon}{N} - \text{plim} \left(\frac{\varepsilon'X}{N} \right) \text{plim} \left(\frac{X'X}{N} \right) \text{plim} \left(\frac{X'\varepsilon}{N} \right) \\ &= \text{plim} \frac{\varepsilon'\varepsilon}{N} - \mathbf{0} \times Q \times \mathbf{0} = \text{plim} \frac{\varepsilon'\varepsilon}{N} \end{aligned}$$

On the other hand,

$$\text{plim} \left(\frac{1}{N-k} \right) \sum_{i=1}^N \varepsilon_i^2 = \text{plim} \left(\frac{N}{N-k} \right) \text{plim} \frac{1}{N} \sum_{i=1}^N \varepsilon_i^2 = \text{plim} \frac{1}{N} \sum_{i=1}^N \varepsilon_i^2 = \text{plim} \frac{\varepsilon'\varepsilon}{N}$$

Therefore, consistency requires that

$$\text{plim} \frac{\varepsilon'\varepsilon}{N} = \sigma^2$$

4. GLS for SURE system is

$$\hat{\beta}_{GLS} = \begin{bmatrix} \sigma^{11}X'_1X_1 & \sigma^{12}X'_1X_2 & \dots & \sigma^{1m}X'_1X_m \\ \sigma^{21}X'_2X_1 & \sigma^{22}X'_2X_2 & \dots & \sigma^{2m}X'_2X_m \\ \dots & \dots & \dots & \dots \\ \sigma^{m1}X'_mX_1 & \sigma^{m2}X'_mX_2 & \dots & \sigma^{mm}X'_mX_m \end{bmatrix}^{-1} \begin{bmatrix} \sum_{j=1}^m \sigma^{1j}X'_1y_j \\ \sum_{j=1}^m \sigma^{2j}X'_2y_j \\ \dots \\ \sum_{j=1}^m \sigma^{mj}X'_my_j \end{bmatrix}$$

If $\sigma_{ij}^2 = 0$ for $i \neq j$, then

$$\begin{aligned} \hat{\beta}_{GLS} &= \begin{bmatrix} \sigma^{11}X'_1X_1 & 0 & \dots & 0 \\ 0 & \sigma^{22}X'_2X_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma^{mm}X'_mX_m \end{bmatrix}^{-1} \begin{bmatrix} \sigma^{11}X'_1y_1 \\ \sigma^{22}X'_2y_2 \\ \dots \\ \sigma^{mm}X'_my_m \end{bmatrix} \\ &= \begin{bmatrix} (X'_1X_1)^{-1}X'_1y_1 \\ (X'_2X_2)^{-1}X'_2y_2 \\ \dots \\ (X'_mX_m)^{-1}X'_my_m \end{bmatrix} = \hat{\beta}_{OLS} \end{aligned}$$

i.e., GLS is equivalent to OLS for individual equations.

If $X_1 = X_2 = \dots = X_{m-1} = X_m$,

$$\begin{aligned} \hat{\beta}_{GLS} &= (X'V^{-1}X)^{-1}X'V^{-1}y \\ &= \left[(I \otimes \bar{X})' (V^{-1} \otimes I) (I \otimes \bar{X}) \right]^{-1} (I \otimes \bar{X})' (V^{-1} \otimes I) y \\ &= \left(V^{-1} \otimes \bar{X}'\bar{X} \right)^{-1} (I \otimes \bar{X})' (V^{-1} \otimes I) y \\ &= \left[V \otimes \left(\bar{X}'\bar{X} \right)^{-1} \right] (I \otimes \bar{X})' (V^{-1} \otimes I) y \\ &= \left[I \otimes \left(\bar{X}'\bar{X} \right)^{-1} \bar{X}' \right] y = \hat{\beta}_{OLS} \end{aligned}$$

recall the product and inverse rules of Kronecker product.

a)

Let $B(L) = \frac{0.6+2L}{1-0.6L+0.5L^2}$. Then, we can write $B(L)$ as :

$$B(L) = \alpha_0 + \alpha_1 L + \alpha_2 L^2 + \alpha_3 L^3 + \dots$$

Multiply both sides by $1 - 0.6L + 0.5L^2$, so

$$(\alpha_0 + \alpha_1 L + \alpha_2 L^2 + \alpha_3 L^3 + \dots)(1 - 0.6L + 0.5L^2) = 0.6 + 2L$$

Since both sides of the equation are equal, it follows that ,

$$\begin{aligned}\alpha_0 &= 0.6 \\ \alpha_1 - 0.6\alpha_0 &= 2 \\ \alpha_j - 0.6\alpha_{j-1} + 0.5\alpha_{j-2} &= 0 \quad \text{for } j = 2, 3, \dots\end{aligned}$$

For the specified coefficients, we can find recursively that

$$\begin{aligned}\alpha_0 &= 0.6 \\ \alpha_1 &= 2.36 \\ \alpha_2 &= 1.266 \\ \alpha_3 &= -0.4204 \\ \alpha_4 &= -0.8852\end{aligned}$$

b) Just do the following: multiply both sides by the denominator

$$\begin{aligned}(1 - \delta L - \eta L^2)y_t &= \alpha(1 - \delta - \eta) + (\beta + \gamma L)x_t + (1 - \delta L - \eta L^2)\varepsilon_t \\ y_t &= \delta y_{t-1} + \eta y_{t-2} + \alpha(1 - \delta - \eta) + \beta x_t + \gamma x_{t-1} + \varepsilon_t - \delta \varepsilon_{t-1} - \eta \varepsilon_{t-2}\end{aligned}$$

This model cannot be estimated consistently using OLS because lags of the dependent variable appear on the LHS, i.e. as regressors, and these will be correlated with the errors. To obtain consistent estimators we need to use the instrumental variable method, and here we could use lags of x to estimate the lagged dependent variable and use the predictors in the original regression.