Cornell University
Department of Economics

Econon 620 - Spring 2004
Instructor: Prof. Kiefer

Problem set # 3

1. (Midterm, 1997) Consider the least squares residual vector e from the regression of y on X, where $V(y) = \sigma^2 I$. Show that the variance of any element of e, say e_j, is less that or equal to σ^2.

2. (Midterm, 199?) Consider the OLS estimator in the model $y = X\beta + \varepsilon$ with $E(\varepsilon) = 0$ and $E(\varepsilon\varepsilon') = \sigma^2 I$. Let the first column of X consist of ones and let the other regressors be measured in deviations from means. Show that the estimator of the intercept is uncorrelated with the estimators of the slopes.

3. Consider the classical multiple regression model;

$$y = X\beta + \varepsilon$$

(a) Show that if $E(\varepsilon) \neq 0$, the least squares estimator is biased.

(b) Show that if we write the model as

$$y = X_1\beta_1 + X_2\beta_2 + \varepsilon$$

and $E(\varepsilon) = X_1\gamma$, the least squares estimator for β_2 is unbiased.

4. Suppose that

$$X_n = 3 - \frac{1}{n^2}$$
$$Y_n = \sqrt{n}\frac{\overline{Z}_n}{\sigma}$$

where $\overline{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ and $Z_i's$ are i.i.d. with mean zero and variance σ^2. Find the limiting distributions of

(a) $X_n + Y_n$ (b) X_nY_n (c) Y_n^2

5. Consider the two regression models;

$$y = \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + \alpha_4 D_4 + \varepsilon$$
$$y = \alpha + \alpha_2 D_2 + \alpha_3 D_3 + \alpha_4 D_4 + \varepsilon$$

The variables are quarterly dummies. There are equal numbers of observations in each quarter. Obtain the least squares estimator in each model. Prove that if there were a term βx in the two equations, the least squares estimators of β would be identical. What happens to if you run the following regression?

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + \alpha_4 D_4 + \varepsilon$$

6. Consider the following regression model;

$$y = X\beta + \varepsilon$$

Assume that

$$\text{plim}\frac{X'X}{N} = Q \text{ where } Q \text{ is positive definite}$$

and $E(\varepsilon) = 0, E(\varepsilon\varepsilon') = \sigma^2 I$. We also assume that X is non-stochastic.

(a) Prove that $\widehat{\beta} \xrightarrow{p} \beta$.

(b) Find the asymptotic distribution of $\sqrt{N}\left(\widehat{\beta} - \beta\right)$.

(c) Prove that plim$s^2 = \sigma^2$ where $s^2 = \frac{e'e}{N-k}$

We will be very careful in indicating which theorem we use in each step. We start from the definition of the least squares estimator;

$$\widehat{\beta} = (X'X)^{-1} X'y = \beta + (X'X)^{-1} X'\varepsilon \tag{1}$$

(a) It is much easier to see what is going on if we express the matrix expression in terms of summation. After a thoughtful moment, you notice that it is given by

$$(X'X) = \sum_{i=1}^{N} x_i x_i'$$

where x_i is a $(k \times 1)$ vector corresponding to the i^{th} observation. From the condition given in the question

$$\text{plim} \frac{1}{N} X'X = Q$$

We can conclude that

$$\text{plim} \frac{1}{N} \sum_{i=1}^{N} x_i x_i' = Q$$

The matrix notation is exactly the condition;

$$\text{plim} \frac{X'X}{N} = Q \tag{2}$$

What about $(X'\varepsilon)$? $-$ remember that $(X'\varepsilon)$ is a $(k \times 1)$ vector $-$. Again it is given by

$$\sum_{i=1}^{N} x_i \varepsilon_i$$

Let's scale the sum by N to get $\frac{1}{N} \sum_{i=1}^{N} x_i \varepsilon_i$. Note that

$$\frac{1}{N} \sum_{i=1}^{N} x_i \varepsilon_i = \frac{1}{N} (x_1 \varepsilon_1 + x_2 \varepsilon_2 + \cdots + x_N \varepsilon_N)$$

The term is the sample average of $x_i \varepsilon_i$, where $x_i \varepsilon_i' s$ are independent random vectors with mean 0 and variance $\sigma^2 x_i x_i'$ since

$$E(x_i \varepsilon_i) = x_i E(\varepsilon_i) = 0 \text{ since } x_i \text{ is non-stochastic.}$$
$$Var(x_i \varepsilon_i) = E(x_i \varepsilon_i \varepsilon_i x_i') = x_i x_i' E(\varepsilon_i^2) = \sigma^2 x_i x_i'$$
$$Cov(x_i \varepsilon_i, x_t \varepsilon_t) = E[x_i \varepsilon_i \varepsilon_t x_t'] = x_i x_t' E(\varepsilon_i \varepsilon_t) = 0 \text{ since } i \neq t.$$

Then, from the Weak Law of Large Numbers(WLLN), we have

$$\frac{1}{N} \sum_{i=1}^{N} x_i \varepsilon_i \xrightarrow{p} \mathbf{0}$$

Then, in vector notation, we have

$$\frac{1}{N}X'\varepsilon \xrightarrow{p} \mathbf{0} \tag{3}$$

We will slightly reshape (1) to get;

$$\widehat{\beta} = \beta + \left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{N}$$

Then,

$$\begin{aligned}
\text{plim}\widehat{\beta} &= \text{plim}\beta + \text{plim}\left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{N} \text{ by (b) in question2} \\
&= \beta + \text{plim}\left(\frac{X'X}{N}\right)^{-1}\text{plim}\frac{X'\varepsilon}{N} \text{ by (a) in question2} \\
&= \beta + \left(\text{plim}\frac{X'X}{N}\right)^{-1}\text{plim}\frac{X'\varepsilon}{N} \text{ by Slutsky's theorem} \\
&= \beta + Q^{-1}\mathbf{0} \text{ by (2) and (3) and } Q \text{ is invertible} \\
&= \beta
\end{aligned}$$

i.e.

$$\widehat{\beta} \xrightarrow{p} \beta$$

In words, the least squares estimator $\widehat{\beta}$ is a consistent estimator for β.

(b) From (1), we have

$$\widehat{\beta} - \beta = (X'X)^{-1}X'\varepsilon$$

Now, we want to scale slightly differently to invoke the Central Limit Theorem(CLT);

$$\sqrt{N}\left(\widehat{\beta} - \beta\right) = \left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{\sqrt{N}} \tag{4}$$

We know that

$$\left(\frac{X'X}{N}\right)^{-1} \xrightarrow{p} Q^{-1} \tag{5}$$

from (2). Now let's take care of $\frac{X'\varepsilon}{\sqrt{N}}$. Again, $\frac{X'\varepsilon}{\sqrt{N}}$ is given by

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}x_i\varepsilon_i = \frac{1}{\sqrt{N}}\left(x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_N\varepsilon_N\right)$$

As we've already seen in (a), $x_i\varepsilon_i's$ are independent random vectors with mean 0 and variance $\sigma^2 x_i x_i'$. Then, by CLT - here, we use a version of

CLT in Page 7 of the lecture note since we have different variances across observations-,

$$\left(\sum_{i=1}^{N}\sigma^2 x_i x_i'\right)^{\frac{1}{2}}\sum_{i=1}^{N}x_i\varepsilon_i \xrightarrow{d} N\left(\mathbf{0},I\right) \tag{6}$$

where $\left(\sum_{i=1}^{N}\sigma^2 x_i x_i'\right)^{\frac{1}{2}}$ is a notation for Λ such that $\Lambda^2=\sum_{i=1}^{N}\sigma^2 x_i x_i'$. However, we know that

$$\frac{1}{N}\sum_{i=1}^{N}\sigma^2 x_i x_i' = \sigma^2\frac{1}{N}\sum_{i=1}^{N}x_i x_i' \xrightarrow{p} \sigma^2 Q \tag{7}$$

from part (a). Hence,

$$\left(\frac{1}{N}\sum_{i=1}^{N}\sigma^2 x_i x_i'\right)^{\frac{1}{2}}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}x_i\varepsilon_i \xrightarrow{d} N\left(\mathbf{0},I\right)$$

becomes

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}x_i\varepsilon_i \xrightarrow{d} N\left(0,\sigma^2 Q\right) \tag{8}$$

Then, from (5) and (8) with (b) in question (3), we have

$$\sqrt{N}\left(\widehat{\beta}-\beta\right)=\left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{\sqrt{N}} \xrightarrow{d} N\left(\mathbf{0},Q^{-1}QQ^{-1}\right)=N\left(\mathbf{0},Q^{-1}\right)$$

(c) Note that

$$\begin{aligned}
s^2 &= \frac{e'e}{N-k}=\frac{\varepsilon' M\varepsilon}{N-k} \text{ since } e=M\varepsilon \\
&= \frac{\varepsilon'\left[I-X\left(X'X\right)^{-1}X'\right]\varepsilon}{N-k}=\frac{N}{N-k}\left[\frac{\varepsilon'\varepsilon}{N}-\frac{\varepsilon' X\left(X'X\right)^{-1}X'\varepsilon}{N}\right] \\
&= \frac{N}{N-k}\left[\frac{\varepsilon'\varepsilon}{N}-\frac{\varepsilon' X}{N}\left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{N}\right]
\end{aligned}$$

Now,

$$\begin{aligned}
\text{plim}s^2 &= \text{plim}\frac{N}{N-k}\text{plim}\left[\frac{\varepsilon'\varepsilon}{N} - \frac{\varepsilon'X}{N}\left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{N}\right] \text{ by (a) in question2} \\
&= \text{plim}\frac{N}{N-k}\left[\text{plim}\frac{\varepsilon'\varepsilon}{N} - \text{plim}\frac{\varepsilon'X}{N}\left(\frac{X'X}{N}\right)^{-1}\frac{X'\varepsilon}{N}\right] \text{ by (b) in question2} \\
&= \text{plim}\frac{N}{N-k}\left[\text{plim}\frac{\varepsilon'\varepsilon}{N} - \text{plim}\frac{\varepsilon'X}{N}\text{plim}\left(\frac{X'X}{N}\right)^{-1}\text{plim}\frac{X'\varepsilon}{N}\right] \text{ by (a) in question2} \\
&= \text{plim}\frac{N}{N-k}\left[\text{plim}\frac{\varepsilon'\varepsilon}{N} - \text{plim}\frac{\varepsilon'X}{N}\left(\text{plim}\frac{X'X}{N}\right)^{-1}\text{plim}\frac{X'\varepsilon}{N}\right] \text{ by } Slutsky's \text{ theorem} \\
&= \left[\sigma^2 - \mathbf{0}'Q^{-1}\mathbf{0}\right] = \sigma^2
\end{aligned}$$

since

$$\begin{aligned}
\underset{N\to\infty}{\text{plim}}\frac{N}{N-k} &= 1, \text{plim}\frac{X'\varepsilon}{N} = \mathbf{0} \text{ by (3)} \\
\left(\text{plim}\frac{X'X}{N}\right)^{-1} &= Q^{-1} \text{ by (5)}
\end{aligned}$$

and

$$\frac{\varepsilon'\varepsilon}{N} = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i^2$$

which is again an average of $\varepsilon_i^2{}'s$ whose mean is $E\left(\varepsilon_i^2\right) = \sigma^2$ and $\varepsilon_i^2{}'s$ are independent - we don't need to calculate the variance here since we use a version of WLLN in *Notes 3* on page 3 of the lecture note. Then, by WLLN

$$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i^2 \xrightarrow{p} \sigma^2$$

Therefore,

$$\text{plim}\frac{\varepsilon'\varepsilon}{N} = \sigma^2$$