Econ 620 Fall 2004
Professor N. Kiefer
TA H. Choi

Suggested Solutions to the midterm exam

1.
(a) $(x-y)'(x-y) = x'x + y'y - x'y - y'x = 1 + 1 - 2x'y \geq 0$
$(x+y)'(x+y) = x'x + y'y + x'y + y'x = 1 + 1 + 2x'y \geq 0$ gives the desired formula.
(b) For the arbitrary length x and y, take $x^* = \frac{x}{\sqrt{x'x}}$, $y^* = \frac{y}{\sqrt{y'y}}$ then apply (a) with x^* and y^*.

2.
(a) There are four kinds of tests. (note, C : critical region, $A = \{0,1\}\backslash C$: acceptance region)
$\mathrm{T}_1 : C = \{0\}$ gives $\alpha = 3/4$, $\beta = 3/4$.
$\mathrm{T}_2 : C = \{1\}$ gives $\alpha = 1/4$, $\beta = 1/4$.
$\mathrm{T}_3 : C = \{0,1\}$ gives $\alpha = 1$, $\beta = 0$.
$\mathrm{T}_4 : C = \emptyset$ gives $\alpha = 0$, $\beta = 1$.
Therefore T_2 is the only test that gives $\alpha = 1/4$, so it is the best test available.

(b) $\beta = 1/4$.

(c) Plot the pair (α, β) from T_1 through T_4. Note that T_1 is dominated by T_2 so the attainable tests are the line connecting T_2,T_3,T_4. in $(\alpha$,$\beta)$ plane.

(d) Now we have 16 possible tests (why?). By the Neyman-Pierson lemma, we know the best test is LR test. First, let (a, b) imply $x_1 = a, x_2 = b$ then consider the test with $\alpha = 1/16$.
Since (1,1) has the maximum LR of $(9/16)/(1/16)$, the test with $C = \{(1,1)\}$ gives the best test with $\alpha = 1/16, \beta = 7/16$. (This is the only test with $\alpha = 1/16$.)
$C = \{(0,1)\}$ or $C = \{(1,0)\}$ gives the best test with $\alpha = 3/16, \beta = 13/16$.
$C = \{(1,1),(0,1)\}$ or $C = \{(1,1),(1,0)\}$ gives the best test with $\alpha = 4/16, \beta = 4/16$.
$C = \{(1,0),(0,1)\}$ gives the best test with $\alpha = 6/16, \beta = 10/16$.
$C = \{(1,1),(0,1),(1,0)\}$ is the best test with $\alpha = 7/16, \beta = 1/16$.
$C = \{(0,0)\}$ gives the best test with $\alpha = 9/16, \beta = 15/16$.
$C = \{(1,1),(0,0)\}$ gives the best test with $\alpha = 10/16, \beta = 6/16$.
$C = \{(0,0),(0,1)\}$ or $C = \{(0,0),(1,0)\}$ gives the best test with $\alpha = 12/16, \beta = 12/16$.
$C = \{(1,1),(0,0),(0,1)\}$ or $C = \{(1,1),(0,0),(1,0)\}$ gives the best test with $\alpha = 13/16, \beta = 3/16$.
$C = \{(1,1),(1,0),(0,1)\}$ gives the best test with $\alpha = 15/16, \beta = 9/16$.
Finally two trivial tests $C = \emptyset$ and $C = \{(1,1),(0,1),(1,0),(0,0)\}$ are the best tests with $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$ respectively.
Plot those 16 points on the (α, β) plane and draw the closest convex hull to the origin by connecting the tests and eliminating the dominated tests.
The more observation we have, the more tests are available. Although we could not see it in this example, we generally can improve the power of the test as sample size increases with a given α.

3. Let $\theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\hat{\theta} = \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}$, $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & \\ 1 & x_N \end{pmatrix}$, and $M = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \\ & & & 0 & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$.

The given model uses only the first and the last observation, so let $\tilde{X} = \begin{pmatrix} 1 & x_1 \\ 1 & x_N \end{pmatrix}$ and get $\begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}$ using OLS estimation. But

$$\begin{aligned} \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} &= (\tilde{X}'\tilde{X})^{-1}\tilde{X}' \begin{pmatrix} y_1 \\ y_N \end{pmatrix} \\ &= (X'MX)^{-1} X'My. \text{(check this!)} \end{aligned}$$

Note that the matrix M picks the first and the last observation from X (Also it is idempotent).

(a) Yes, it is unbiased since $E\hat{\theta} = E\begin{pmatrix}\alpha^* \\ \beta^*\end{pmatrix} = E\left(X'MX\right)^{-1}X'My = E\left(X'MX\right)^{-1}X'M(X\theta+\varepsilon) = \theta.$

(b) $\text{Var}(\hat{\theta}) = E(\hat{\theta}-\theta)(\hat{\theta}-\theta)' = \sigma^2(X'MX)^{-1} = \sigma^2\begin{pmatrix}2 & x_1+x_N \\ x_1+x_N & x_1^2+x_N^2\end{pmatrix}^{-1}$

(c) Note that x_1 and x_N goes to negative infinity and positive infinity repectively in probability (or almost surely). Using this we have plim $\text{Var}(\beta^*) = 0$ so, plim $\beta^* = \beta$. But plim $\alpha^* \neq \alpha$. (check this !). Since this problem can be interpreted as we have to fix x_1 and x_N, I gave the full credit to those who solved this using fixed x_1 and x_N, and obtained the result that the both estimators are inconsistent.

(d) Let $A_1 = (1\ 1\ ...1\ 0\ 0\ ...\ 0)'$ and $A_2 = (0\ 0\ ...0\ 1\ 1\ ...\ 1)'$ that is A_1 has $N/2$ ones in the first half elements and A_2 has them in the second half. If we define P_{A_1} and P_{A_2} as the projection onto the span of A_1 and A_2 respectively, $P_{A_1}+P_{A_2}$ transforms a vector into $N/2$ elements of the mean of the first half observation and $N/2$ elements of the mean of the rest half observtion. For example,

$$
\begin{aligned}
(P_{A_1}+P_{A_2})y &= P_{A_1}y + P_{A_2}y \\
&= \begin{pmatrix} y(1) \\ y(1) \\ \vdots \\ y(1) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y(2) \\ y(2) \\ \vdots \\ y(2) \end{pmatrix} = \begin{pmatrix} y(1) \\ y(1) \\ \vdots \\ y(1) \\ y(2) \\ y(2) \\ \vdots \\ y(2) \end{pmatrix}. \text{ (check this!)}
\end{aligned}
$$

The estimation of the given model is based on $\tilde{X} = \begin{pmatrix}1 & x(1) \\ 1 & x(2)\end{pmatrix}$ but using the fact that $P_{A_1}+P_{A_2}$ is idempotent (why? - use A_1 and A_2 are orthogonal), we have

$$
\begin{aligned}
\begin{pmatrix}\alpha^{**} \\ \beta^{**}\end{pmatrix} &= (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\begin{pmatrix}y(1) \\ y(2)\end{pmatrix} = \left(\frac{X'(P_{A_1}+P_{A_2})X}{N/2}\right)^{-1}\frac{X'(P_{A_1}+P_{A_2})y}{N/2}. \\
&= \left(X'(P_{A_1}+P_{A_2})X\right)^{-1}X'(P_{A_1}+P_{A_2})y. \text{(check this!)}
\end{aligned}
$$

Now we have the unbiasedness of the estimators since

$$
\begin{aligned}
E\left(X'(P_{A_1}+P_{A_2})X\right)^{-1}X'(P_{A_1}+P_{A_2})y &= E\left(X'(P_{A_1}+P_{A_2})X\right)^{-1}X'(P_{A_1}+P_{A_2})(X\theta+\varepsilon) \\
&= \theta.
\end{aligned}
$$

(e)

$$
\begin{aligned}
Var(\hat{\theta}) &= \sigma^2\left(X'(P_{A_1}+P_{A_2})X\right)^{-1} \\
&= \sigma^2\left[\frac{N}{2}\begin{pmatrix}1 & x(1) \\ 1 & x(2)\end{pmatrix}'\begin{pmatrix}1 & x(1) \\ 1 & x(2)\end{pmatrix}\right]^{-1} \\
&= \frac{2\sigma^2}{N}\begin{pmatrix}2 & x(1)+x(2) \\ x(1)+x(2) & x(1)^2+x(2)^2\end{pmatrix}^{-1}
\end{aligned}
$$

(f) Both estimators are consistent. Note that $x(1)$ $x(2)$, $y(1)$, $y(2)$ converges to $E(x(1))$, $E(x(2))$, $E(y(1))$, $E(y(2))$ respectively. What is those values? Try to get that by assuming a density of x_i.

(g) Clearly $\hat{\theta} = \left(X'(P_{A_1}+P_{A_2})X\right)^{-1}X'(P_{A_1}+P_{A_2})y$ is linear in y, therefore Gauss-Markov theorem tells us that it is not BLUE.

Note that we can simply use

$$
\begin{aligned}
\begin{pmatrix}\alpha^* \\ \beta^*\end{pmatrix} &= \begin{pmatrix}1 & x_1 \\ 1 & x_N\end{pmatrix}^{-1}\begin{pmatrix}y_1 \\ y_N\end{pmatrix} \\
\begin{pmatrix}\alpha^{**} \\ \beta^{**}\end{pmatrix} &= \begin{pmatrix}1 & x(1) \\ 1 & x(2)\end{pmatrix}^{-1}\begin{pmatrix}y(1) \\ y(2)\end{pmatrix}
\end{aligned}
$$

for estimation. But the above solutions illustrate the generality and the usefulness of the matrix form.