Econ 620
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Suggested Solution to the Midterm Exam

1. (Warmup) First of all, note that

$$Z'Z = \begin{bmatrix} y' \\ X' \end{bmatrix} \begin{bmatrix} y & X \end{bmatrix} = \begin{bmatrix} y'y & y'X \\ X'y & X'X \end{bmatrix}$$

Now, the OLS estimate is given by;

$$\widehat{\beta} = (X'X)^{-1} X'y = \begin{bmatrix} 20 & 0 \\ 0 & 75 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 25 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

To find s^2, we have to find $e'e$;

$$\begin{aligned} e'e &= \left(y - X\widehat{\beta}\right)' \left(y - X\widehat{\beta}\right) = y'y - \widehat{\beta}' X'X\widehat{\beta} \\ &= [100] - \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 75 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix} = \frac{260}{3} = 86.667 \end{aligned}$$

Then, from the formula for s^2;

$$s^2 = \frac{e'e}{N-k} = \frac{\frac{260}{3}}{(20-2)} = \frac{130}{27} = 4.8148$$

Finally, we have

$$R^2 = 1 - \frac{e'e}{y'Ay}$$

Note that

$$\begin{aligned} y'Ay &= y'\left[I - \mathbf{1}\left(\mathbf{1}'\mathbf{1}\right)^{-1}\mathbf{1}'\right]y = y'y - \frac{1}{N}y'\mathbf{1}\mathbf{1}'y \\ &= y'y - N\overline{y}^2 = 100 - 20 \times \left(\frac{10}{20}\right)^2 = 95 \end{aligned}$$

Then,

$$R^2 = 1 - \frac{\frac{130}{27}}{95} = \frac{487}{513} = 0.94932$$

The full data set is a (20×3) matrix whose first column is the vector of observations on the dependent variable and the remaining columns are the vector of ones corresponding to the constant term and vector of observations on the independent variable. On the other hand, the cross product matrix given in the question is simply a (3×3) matrix. Still, we can extract every bit of information we need from the cross product matrix only. We do not lose anything just by looking at the cross product matrix. In other words, the cross product matrix is a sufficient statistic for the regression model.

2. To get $\widetilde{\beta}$, we follow the instruction in the question;

$$\begin{aligned} \overline{y}(1) &= \widetilde{\beta}_1 + \widetilde{\beta}_2 \\ \overline{y}(2) &= \widetilde{\beta}_1 - \widetilde{\beta}_2 \end{aligned}$$

Then,

$$\left[\widetilde{\beta}\right] = \begin{bmatrix} \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\overline{y}(1) + \overline{y}(2)\right) \\ \frac{1}{2}\left(\overline{y}(1) - \overline{y}(2)\right) \end{bmatrix}$$

To check the unbiasedness, we will transform our estimator slightly;

$$\begin{aligned} \begin{bmatrix} \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}\left(\left(\beta_1 + \beta_2 + \overline{\epsilon}(1)\right) + \left(\beta_1 - \beta_2 + \overline{\epsilon}(2)\right)\right) \\ \frac{1}{2}\left(\left(\beta_1 + \beta_2 + \overline{\epsilon}(1)\right) - \left(\beta_1 - \beta_2 + \overline{\epsilon}(2)\right)\right) \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 + \frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2) \\ \beta_2 + \frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2) \end{bmatrix} \end{aligned}$$

Then,

$$\begin{bmatrix} E\left(\widetilde{\beta}_1\right) \\ E\left(\widetilde{\beta}_2\right) \end{bmatrix} = \begin{bmatrix} \beta_1 + \frac{1}{2}E\left[\overline{\epsilon}(1)\right] + \frac{1}{2}E\left[\overline{\epsilon}(2)\right] \\ \beta_2 + \frac{1}{2}E\left[\overline{\epsilon}(1)\right] - \frac{1}{2}E\left[\overline{\epsilon}(2)\right] \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

since $E[\overline{\epsilon}(j)] = 0, j = 1, 2$. Therefore, $\widetilde{\beta}$ is unbiased. As for variance matrix,

$$Var\left[\widetilde{\beta}\right] = E\left[\left(\widetilde{\beta} - E\left(\widetilde{\beta}\right)\right)\left(\widetilde{\beta} - E\left(\widetilde{\beta}\right)\right)'\right] = E\left[\left(\widetilde{\beta} - \beta\right)\left(\widetilde{\beta} - \beta\right)'\right]$$

$$= E\left[\begin{bmatrix} \frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2) \\ \frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2) \end{bmatrix} \begin{bmatrix} \frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2) & \frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2) \end{bmatrix}\right]$$

$$= E\begin{bmatrix} \left(\frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2)\right)^2 & \left(\frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2)\right)\left(\frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2)\right) \\ \left(\frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2)\right)\left(\frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2)\right) & \left(\frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2)\right)^2 \end{bmatrix}$$

We now calculate the expectation of each element;

$$E\left[\left(\frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2)\right)^2\right] = \frac{1}{4}\left\{E\left[(\overline{\epsilon}(1))^2\right] + E\left[(\overline{\epsilon}(2))^2\right] + 2E\left[(\overline{\epsilon}(1))(\overline{\epsilon}(2))\right]\right\}$$

$$= \frac{1}{4}\left[Var(\overline{\epsilon}(1)) + Var(\overline{\epsilon}(2)) + 2Cov(\overline{\epsilon}(1), \overline{\epsilon}(2))\right]$$

$$= \frac{1}{4}\left[\frac{2}{N}\sigma^2 + \frac{2}{N}\sigma^2\right] = \frac{1}{N}\sigma^2$$

since $E(\overline{\epsilon}(j)) = 0$, $Var(\overline{\epsilon}(j)) = \frac{2}{N}\sigma^2, j = 1, 2.$, and $Cov(\overline{\epsilon}(1), \overline{\epsilon}(2)) = 0-$ Verify these!-. Continuing the calculation, we have

$$E\left[\left(\frac{1}{2}\overline{\epsilon}(1) + \frac{1}{2}\overline{\epsilon}(2)\right)\left(\frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2)\right)\right] = \frac{1}{4}E\left[\overline{\epsilon}(1)^2 - \overline{\epsilon}(2)^2\right]$$

$$= \frac{1}{4}\left[Var(\overline{\epsilon}(1)) - Var(\overline{\epsilon}(2))\right] = 0$$

Finally,

$$E\left[\left(\frac{1}{2}\overline{\epsilon}(1) - \frac{1}{2}\overline{\epsilon}(2)\right)^2\right] = \frac{1}{4}\left\{E\left[(\overline{\epsilon}(1))^2\right] + E\left[(\overline{\epsilon}(2))^2\right] - 2E\left[(\overline{\epsilon}(1))(\overline{\epsilon}(2))\right]\right\}$$

$$= \frac{1}{4}\left[Var(\overline{\epsilon}(1)) + Var(\overline{\epsilon}(2)) - 2Cov(\overline{\epsilon}(1), \overline{\epsilon}(2))\right]$$

$$= \frac{1}{4}\left[\frac{2}{N}\sigma^2 + \frac{2}{N}\sigma^2\right] = \frac{1}{N}\sigma^2$$

Therefore,

$$Var\left(\widetilde{\beta}\right) = \sigma^2\begin{bmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{bmatrix}$$

To check the consistency;

$$\begin{bmatrix} \text{plim}\widetilde{\beta}_1 \\ \text{plim}\widetilde{\beta}_2 \end{bmatrix} = \begin{bmatrix} \beta_1 + \text{plim}\frac{1}{2}\overline{\epsilon}(1) + \text{plim}\frac{1}{2}\overline{\epsilon}(2) \\ \beta_2 + \text{plim}\frac{1}{2}\overline{\epsilon}(1) - \text{plim}\frac{1}{2}\overline{\epsilon}(2) \end{bmatrix}$$

To make the presentation simple, let's arrange the data so that we have $\frac{N}{2}$ observations from the Group 1 first and $\frac{N}{2}$ observations from the Group 2 later.

$$\begin{aligned} \text{plim}\frac{1}{2}\overline{\epsilon}(1) &= \frac{1}{2}\text{plim}\overline{\epsilon}(1) = \frac{1}{2}\text{plim}\frac{1}{\left(\frac{N}{2}\right)}\sum_{i=1}^{\frac{N}{2}}\epsilon_i = \frac{1}{2}E(\epsilon_i) = 0 \\ \text{plim}\frac{1}{2}\overline{\epsilon}(2) &= \frac{1}{2}\text{plim}\overline{\epsilon}(2) = \frac{1}{2}\text{plim}\frac{1}{\left(\frac{N}{2}\right)}\sum_{i=\frac{N}{2}+1}^{N}\epsilon_i = \frac{1}{2}E(\epsilon_i) = 0 \end{aligned}$$

Hence,

$$\text{plim}\widetilde{\beta} = \beta$$

i.e., $\widetilde{\beta}$ is a consistent estimator for β. A little bit easier way to prove the consistency is to notice the fact that

$$E\left(\widetilde{\beta}\right) = \beta, \quad \lim_{N\to\infty} Var\left(\widetilde{\beta}\right) = 0$$

implies that $\widetilde{\beta} \xrightarrow{m.s.} \beta$, which consequently implies that $\widetilde{\beta} \xrightarrow{p} \beta$.

What is the OLS estimator then? First of all, note that the X matrix is given by

$$X = \begin{bmatrix} 1 & 1 \\ .. & .. \\ 1 & 1 \\ 1 & -1 \\ .. & .. \\ 1 & -1 \end{bmatrix}$$

Again, we record $\frac{N}{2}$ observations from the Group 1 first and $\frac{N}{2}$ observations from the Group 2 later. Then,

$$X'X = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \text{ and } X'y = \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{\frac{N}{2}} y_i - \sum_{i=\frac{N}{2}+1}^{N} y_i \end{bmatrix}$$

Hence,

$$\begin{aligned}
\widehat{\beta}_{OLS} &= (X'X)^{-1}(X'y) = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{N} y_i \\ \sum_{i=1}^{\frac{N}{2}} y_i - \sum_{i=\frac{N}{2}+1}^{N} y_i \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{N}\left(\sum_{i=1}^{\frac{N}{2}} y_i + \sum_{i=\frac{N}{2}+1}^{N} y_i\right) \\ \frac{1}{N}\left(\sum_{i=1}^{\frac{N}{2}} y_i - \sum_{i=\frac{N}{2}+1}^{N} y_i\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{2}{N}\sum_{i=1}^{\frac{N}{2}} y_i\right) + \frac{1}{2}\left(\frac{2}{N}\sum_{i=\frac{N}{2}+1}^{N} y_i\right) \\ \frac{1}{2}\left(\frac{2}{N}\sum_{i=1}^{\frac{N}{2}} y_i\right) - \frac{1}{2}\left(\frac{2}{N}\sum_{i=\frac{N}{2}+1}^{N} y_i\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\left[\overline{y}(1) + \overline{y}(2)\right] \\ \frac{1}{2}\left[\overline{y}(1) - \overline{y}(2)\right] \end{bmatrix}
\end{aligned}$$

Therefore,

$$\widehat{\beta}_{OLS} = \widetilde{\beta}$$

We can infer that

$$Var\left(\widehat{\beta}_{OLS}\right) = Var\left(\widetilde{\beta}\right) = \sigma^2 \begin{bmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{bmatrix}$$

Or you can calculate the variance matrix with;

$$Var\left(\widehat{\beta}_{OLS}\right) = \sigma^2 (X'X)^{-1} = \sigma^2 \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{bmatrix}$$

which is exactly what we expected.

3. We assume that

$$x_i \sim i.i.d.\left(\mu_X, \sigma_X^2\right) \text{ and } \epsilon_i \sim i.i.d.\left(0, \sigma_\epsilon^2\right)$$

The OLS estimator is given by;

$$\begin{aligned}
\widehat{\beta} &= \frac{\sum_{i=1}^{n}(z_i - \overline{z})(y_i - \overline{y})}{\sum_{i=1}^{n}(z_i - \overline{z})^2} = \frac{\sum_{i=1}^{n}(z_i - \overline{z})\, y_i}{\sum_{i=1}^{n}(z_i - \overline{z})^2} = \frac{\sum_{i=1}^{n}(z_i - \overline{z})(\alpha + \beta x_i + \epsilon_i)}{\sum_{i=1}^{n}(z_i - \overline{z})^2} \\
&= \frac{\alpha \sum_{i=1}^{n}(z_i - \overline{z})}{\sum_{i=1}^{n}(z_i - \overline{z})^2} + \frac{\beta \sum_{i=1}^{n}(z_i - \overline{z})\, x_i}{\sum_{i=1}^{n}(z_i - \overline{z})^2} + \frac{\sum_{i=1}^{n}(z_i - \overline{z})\, \epsilon_i}{\sum_{i=1}^{n}(z_i - \overline{z})^2} \\
&= \frac{\beta \sum_{i=1}^{n}(z_i - \overline{z})\, x_i}{\sum_{i=1}^{n}(z_i - \overline{z})^2} + \frac{\sum_{i=1}^{n}(z_i - \overline{z})\, \epsilon_i}{\sum_{i=1}^{n}(z_i - \overline{z})^2} \text{ since } \sum_{i=1}^{n}(z_i - \overline{z}) = 0
\end{aligned}$$

Then,

$$\text{plim}\widehat{\beta} = \beta \frac{\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})\, x_i}{\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2} + \frac{\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})\, \epsilon_i}{\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2}$$

We further assume that

$$\nu_i \sim i.i.d. \left(0, \sigma_\nu^2\right)$$

and ν_i is independent of x_i and ϵ_i. Let's calculate probability limits;

$$\begin{aligned}
\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})^2 &= \text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i + \nu_i - \overline{x} - \overline{\nu})^2 = \text{plim}\frac{1}{n}\sum_{i=1}^{n}\left[(x_i - \overline{x}) + (\nu_i - \overline{\nu})\right]^2 \\
&= \text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 + 2\text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})(\nu_i - \overline{\nu}) \\
&\quad + \text{plim}\frac{1}{n}\sum_{i=1}^{n}(\nu_i - \overline{\nu})^2 \\
&= \sigma_X^2 + 0 + \sigma_\nu^2 = \sigma_X^2 + \sigma_\nu^2 \text{ since } x_i \text{ and } \nu_i \text{ is independent}
\end{aligned}$$

$$\begin{aligned}
\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})x_i &= \text{plim}\frac{1}{n}\sum_{i=1}^{n}\left[(x_i - \overline{x}) + (\nu_i - \overline{\nu})\right]x_i \\
&= \text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})x_i + \text{plim}\frac{1}{n}\sum_{i=1}^{n}(\nu_i - \overline{\nu})x_i \\
&= \text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 + \text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})(\nu_i - \overline{\nu}) \\
&= \sigma_X^2
\end{aligned}$$

On the other hand;

$$\begin{aligned}
\text{plim}\frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})\epsilon_i &= \text{plim}\frac{1}{n}\sum_{i=1}^{n}\left[(x_i - \overline{x}) + (\nu_i - \overline{\nu})\right]\epsilon_i \\
&= \text{plim}\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})\epsilon_i + \text{plim}\frac{1}{n}\sum_{i=1}^{n}(\nu_i - \overline{\nu})\epsilon_i \\
&= 0 \text{ since } \epsilon_i \text{ is independent of } x_i \text{ and } \nu_i
\end{aligned}$$

Therefore,

$$\text{plim}\widehat{\beta} = \beta\frac{\sigma_X^2}{\sigma_X^2 + \sigma_\nu^2} + \frac{0}{\sigma_X^2 + \sigma_\nu^2} = \beta\frac{\sigma_X^2}{\sigma_X^2 + \sigma_\nu^2} < \beta$$

For the variety, we will derive the probability limit in vector notation. Our model is given by;

$$w_i y_i = \gamma w_i + \delta w_i z_i + u_i$$

The OLS estimator is;

$$\begin{bmatrix} \widehat{\gamma} \\ \widehat{\delta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} w_i^2 & \sum_{i=1}^{n} w_i w_i z_i \\ \sum_{i=1}^{n} w_i w_i z_i & \sum_{i=1}^{n} w_i^2 z_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} w_i w_i y_i \\ \sum_{i=1}^{n} w_i z_i w_i y_i \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} w_i^2 & \sum_{i=1}^{n} w_i^2 z_i \\ \sum_{i=1}^{n} w_i^2 z_i & \sum_{i=1}^{n} w_i^2 z_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} w_i^2 y_i \\ \sum_{i=1}^{n} w_i^2 z_i y_i \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \text{plim}\widehat{\gamma} \\ \text{plim}\widehat{\delta} \end{bmatrix} = \text{plim}\left[\frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n} w_i^2 & \sum_{i=1}^{n} w_i^2 z_i \\ \sum_{i=1}^{n} w_i^2 z_i & \sum_{i=1}^{n} w_i^2 z_i^2 \end{bmatrix}\right]^{-1} \text{plim}\frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n} w_i^2 y_i \\ \sum_{i=1}^{n} w_i^2 z_i y_i \end{bmatrix}$$

$$= \begin{bmatrix} \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 & \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i \\ \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i & \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 y_i \\ \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i y_i \end{bmatrix}$$

Suppose that $w_i \sim i.i.d.\left(\mu_W, \sigma_W^2\right)$ and $E\left(w_i x_i\right) = \sigma_{WX} \neq 0$. Moreover, we assume that w_i is independent of ϵ_i and ν_i.

$$\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 = E\left(w_i^2\right)$$

$$\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i = \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2\left(x_i+\nu_i\right) = \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 x_i + \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \nu_i$$

$$= E\left(w_i^2 x_i\right) - E\left(w_i^2 \nu_i\right) = E\left(w_i^2 x_i\right) - E\left(w_i^2\right)E\left(\nu_i\right) = E\left(w_i^2 x_i\right)$$

$$\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i^2 = \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2\left(x_i+\nu_i\right)^2 = \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 x_i^2$$

$$+2\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \nu_i + \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \nu_i^2$$

$$= E\left(w_i^2 x_i^2\right) + 2E\left(w_i^2 \nu_i\right) + E\left(w_i^2 \nu_i^2\right)$$

$$= E\left(w_i^2 x_i^2\right) + 2\left[E\left(w_i^2\right)E\left(\nu_i\right)\right] + E\left(w_i^2\right)E\left(\nu_i^2\right)$$

$$= E\left(w_i^2 x_i^2\right) + \sigma_\nu^2 E\left(w_i^2\right)$$

And

$$\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 y_i = \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2\left(\alpha+\beta x_i+\epsilon_i\right)$$

$$= \alpha\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 + \beta\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 x_i + \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \epsilon_i$$

$$
\begin{aligned}
&= \alpha E\left(w_i^2\right) + \beta E\left(w_i^2 x_i\right) + E\left(w_i^2 \epsilon_i\right) \\
&= \alpha E\left(w_i^2\right) + \beta E\left(w_i^2 x_i\right) + E\left(w_i^2\right) E\left(\epsilon_i\right) \\
&= \alpha E\left(w_i^2\right) + \beta E\left(w_i^2 x_i\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 z_i y_i &= \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \left(x_i + \nu_i\right)\left(\alpha + \beta x_i + \epsilon_i\right) \\
&= \alpha \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 x_i + \beta \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 x_i^2 + \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 x_i \epsilon_i \\
&\quad + \alpha \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \nu_i + \beta \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \nu_i x_i + \text{plim}\frac{1}{n}\sum_{i=1}^{n} w_i^2 \nu_i \epsilon_i \\
&= \alpha E\left(w_i^2 x_i\right) + \beta E\left(w_i^2 x_i^2\right) + E\left(w_i^2 x_i \epsilon_i\right) \\
&\quad + \alpha E\left(w_i^2 \nu_i\right) + \beta E\left(w_i^2 \nu_i x_i\right) + E\left(w_i^2 \nu_i \epsilon_i\right) \\
&= \alpha E\left(w_i^2 x_i\right) + \beta E\left(w_i^2 x_i^2\right)
\end{aligned}
$$

Therefore,

$$
\begin{bmatrix} \text{plim}\widehat{\gamma} \\ \text{plim}\widehat{\delta} \end{bmatrix} = \begin{bmatrix} E\left(w_i^2\right) & E\left(w_i^2 x_i\right) \\ E\left(w_i^2 x_i\right) & E\left(w_i^2 x_i^2\right) + \sigma_\nu^2 E\left(w_i^2\right) \end{bmatrix}^{-1} \begin{bmatrix} \alpha E\left(w_i^2\right) + \beta E\left(w_i^2 x_i\right) \\ \alpha E\left(w_i^2 x_i\right) + \beta E\left(w_i^2 x_i^2\right) \end{bmatrix}
$$

Then,

$$
\begin{aligned}
\text{plim}\widehat{\delta} &= \frac{-E\left(w_i^2 x_i\right)\left[\alpha E\left(w_i^2\right) + \beta E\left(w_i^2 x_i\right)\right] + E\left(w_i^2\right)\left[\alpha E\left(w_i^2 x_i\right) + \beta E\left(w_i^2 x_i^2\right)\right]}{E\left(w_i^2\right)\left[E\left(w_i^2 x_i^2\right) + \sigma_\nu^2 E\left(w_i^2\right)\right] - \left[E\left(w_i^2 x_i\right)\right]^2} \\
&= \frac{\beta\left[E\left(w_i^2\right)E\left(w_i^2 x_i^2\right) - \left[E\left(w_i^2 x_i\right)\right]^2\right]}{E\left(w_i^2\right)\left[E\left(w_i^2 x_i^2\right) + \sigma_\nu^2 E\left(w_i^2\right)\right] - \left[E\left(w_i^2 x_i\right)\right]^2} \neq \beta \text{ as long as } \sigma_\nu^2 \neq 0.
\end{aligned}
$$